Discrete Mathematic

Chapter 9: Graphs

9.5 Euler and Hamilton Paths

9.7 Planar Graphs

9.8 Graph Coloring

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Agenda

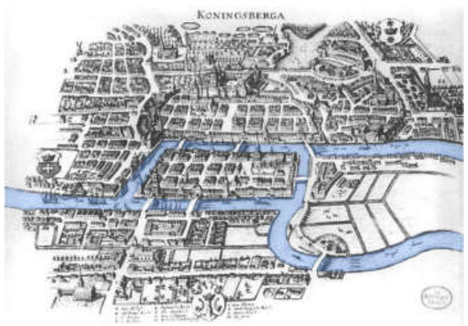
- Euler Path
- Hamilton Path
- Planar Graph
- Coloring

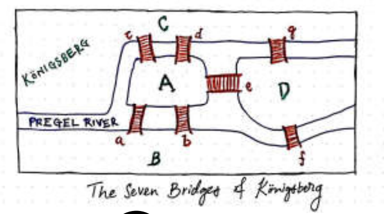
Euler and Hamiltonian Path

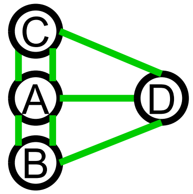
- **Euler Path**
  a path visits every edge exactly once
- **Hamiltonian Path**
  a path visits every vertex exactly once

Euler Path

- **Seven Bridges of Königsberg**
  - Königsberg is built on both banks of the Preger river
  - Now a city in Russia called Kaliningrad
- Is it possible to walk through the city that would cross each of bridges once





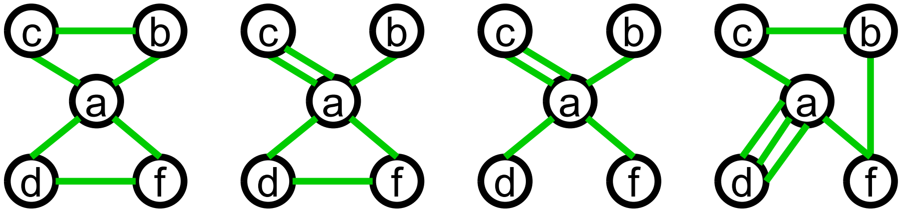


Euler Path

- **Leonhard Euler**, the Swiss mathematician, was also unable to find such a route
- Euler figured out how to show for certain that no such route existed

Euler Path

- Euler Path: a path visits every edge exactly once
- Euler Cycle: Euler path which starts and stops at the same vertex
- A connected graph G is called Eulerian if it contains an Euler path

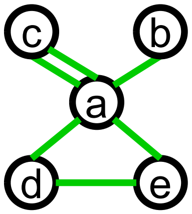


Euler Path	YES	YES	NO	YES
Euler Cycle	YES	NO	NO	NO
Eulerian	YES	YES	NO	YES

Euler Path

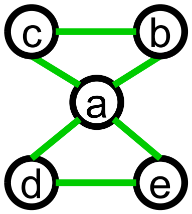
- Observation from an Euler path,

a > c > a > d > e > a > b
Euler Path



Starting / Intermediate	deg(a) = 5	(O)
End	deg(b) = 1	(O)
Intermediate	deg(c) = 2	(E)
Intermediate	deg(d) = 2	(E)
Intermediate	deg(e) = 2	(E)

a > c > b > a > d > f > a
Euler Cycle



deg(a) = 4	(E)	Starting / Intermediate / End
deg(b) = 2	(E)	Intermediate
deg(c) = 2	(E)	Intermediate
deg(d) = 2	(E)	Intermediate
deg(e) = 2	(E)	Intermediate

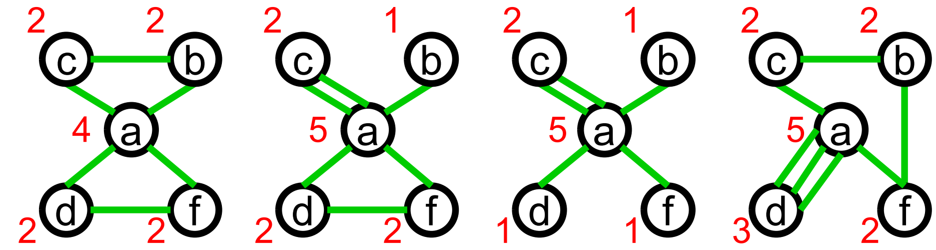
Euler Path

- Observation from an Euler path,
  - Intermediate vertex
    - Degree must be even (Entrance and exist connection)
  - Starting and end vertices
    - If the same (cycle), degree are even
    - If different (non-cycle), degrees are odd (in or out)

Euler Path

- **Theorem 1**
  A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree
- **Theorem 2**
  A connected multigraph has Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree

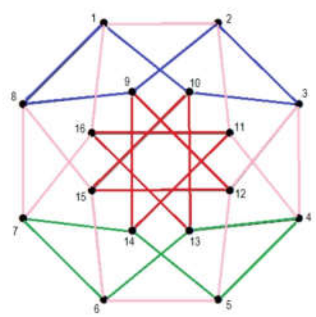
Euler Path



	Graph 1	Graph 2	Graph 3	Graph 4
Euler Path	YES	YES	NO	YES
Euler Cycle	YES	NO	NO	NO
Eulerian	YES	YES	NO	YES
Odd Degree	0	2	4	2
Even Degree	5	3	1	3
		Starting/End Vertices must be a and c		Starting/End Vertices must be a and d

Euler Path

- How to identify an Euler Path / Cycle?
  - Euler Path
    Fleury's Algorithm
  - Euler Cycle
    Hierholzer's Algorithm

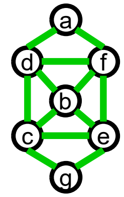


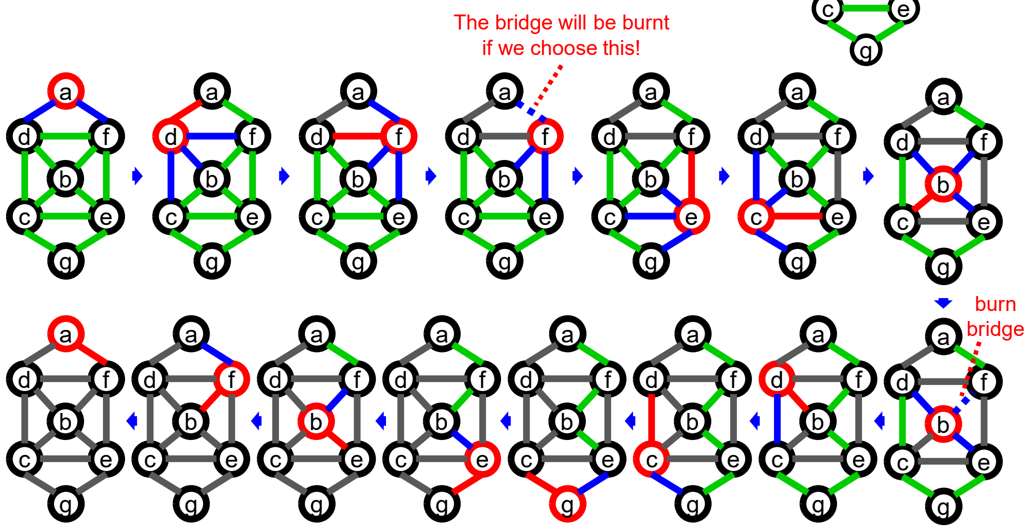
Euler Path

Fleury's Algorithm

- Identify Euler Path
  1. If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them
  2. Follow edges one at a time. If you have a choice between a bridge and a non-bridge, always choose the non-bridge
  3. Stop when you run out of edges
- "Don't burn bridges" so that we can come back to a vertex and traverse remaining edges

Fleury's Algorithm



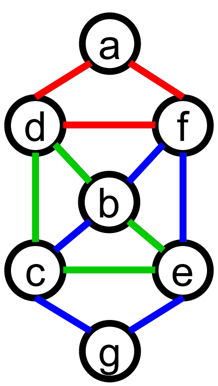


Hierholzer's Algorithm

- Identify Euler Cycle
  1. Select a node v as a starting node
  2. Form a cycle using non-traveled edges and end at v (remove the visited edges)
  3. While all edges have been traversed, stop
     a) Find a node u on the previous cycles that's connected to a non-traveled edge
     b) Form a cycle using non-traveled edges and end at u (remove the visited edges)
     c) Merge both tours at the node u

Hierholzer

1. Select a node v as a starting node
2. Form a cycle using non-traveled edges and end at v (remove the visited edges)
3. While all edges have been traversed, stop
   a) Find a node v on the previous cycles that's connected to a non-traveled edge
   b) Form a cycle using non-traveled edges and end at v (remove the visited edges)
   c) Merge both tours at the node v



a > d > f > a

d > b > e > c > d

f > b > c > g > e > f

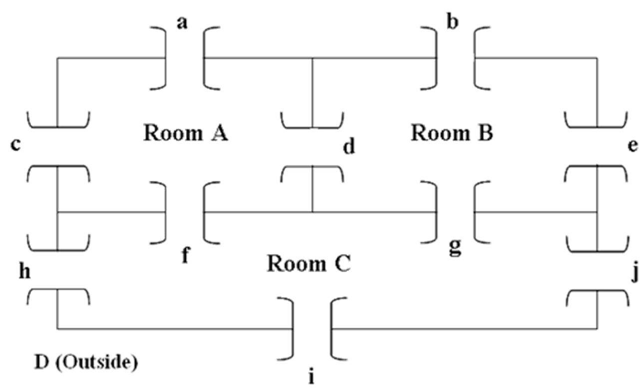
a > d > f > a

d > b > e > c > d   f > b > c > g > e > f

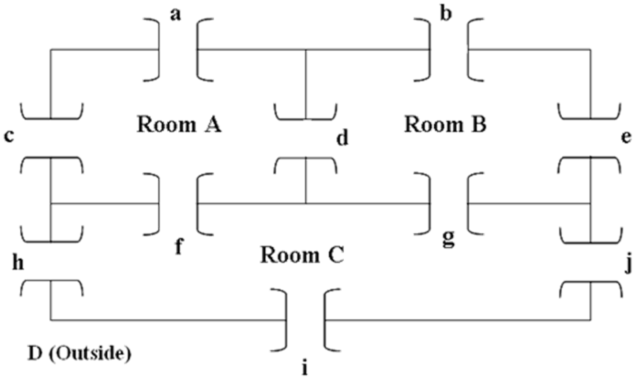
a > d > b > e > c > d > f > b > c > g > e > f > a

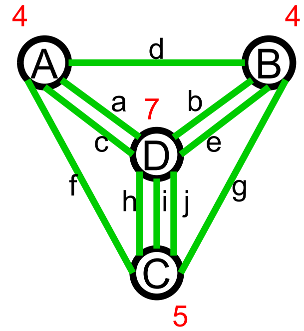
Example 1

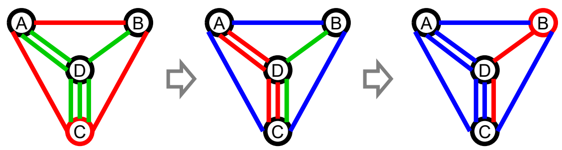
- Is it possible to begin in a room or the outside and take a walk that goes through each door exactly once? If yes, how?



Euler Path
Example 1

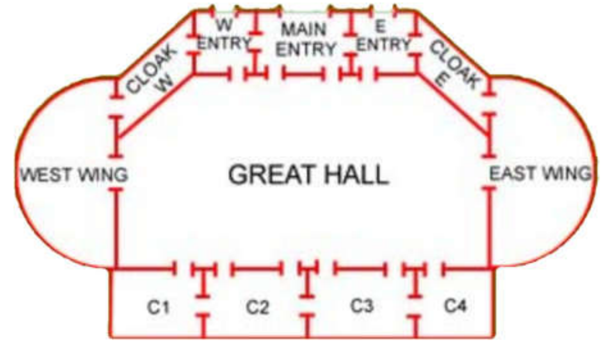




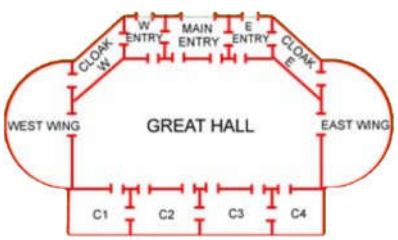


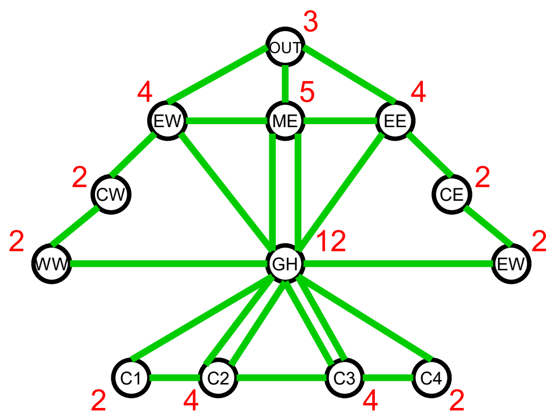
Euler Path
Example 2

- Is it possible to walk through and around this building passing through each and every doorway exactly once?

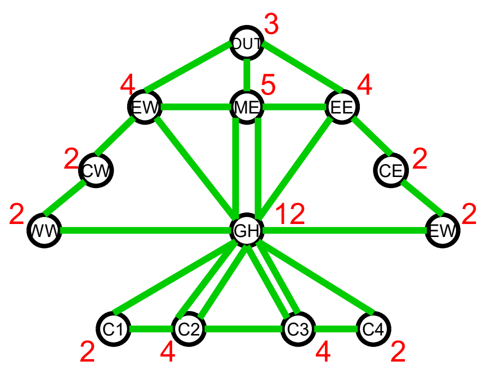


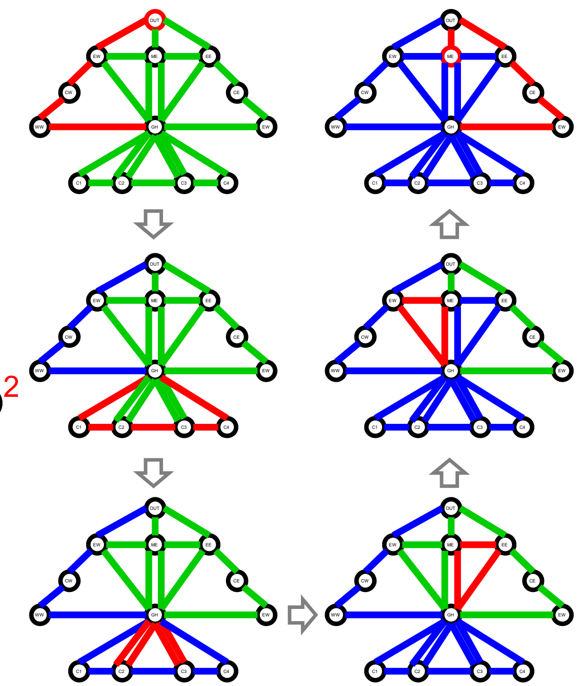
Euler Path
Example 2





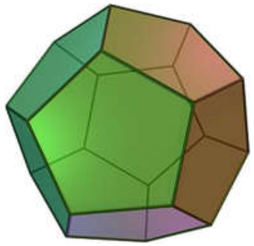
Euler Path
Example 2

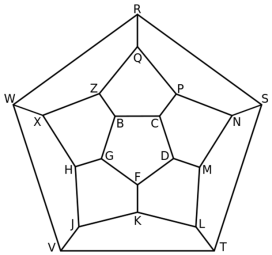




Hamiltonian Path

- **Icosian game**
  - Invented by an Irishman named Sir William Rowan Hamilton (1805-1865)
  - Is there a cycle in the dodecahedron puzzle that passes through each vertex exactly once?

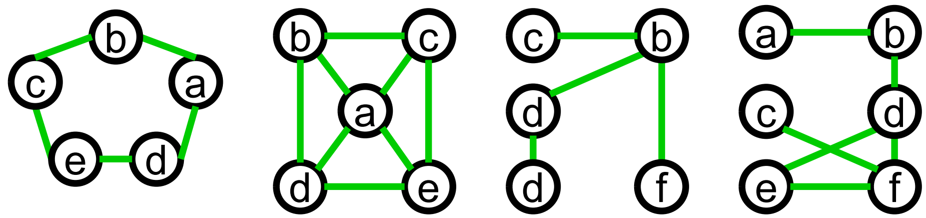




Dodecahedron puzzle

Hamilton Path

- Hamilton Path: a path visits every vertex exactly once
- Hamilton Cycle: Hamilton path which starts and stops at the same vertex
- Self-loop and multiple edges can be ignored

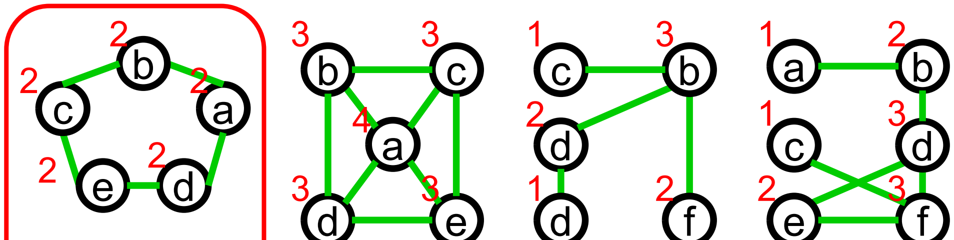


Ham. Path	YES	YES	NO	YES
Ham. Cycle	YES	YES	NO	NO

Hamilton Path

Dirac's Theorem

- **Theorem**: If each vertex of a simple graph with n vertices and $n \geq 3$ has degree $\geq n/2$, there is Hamilton circuit

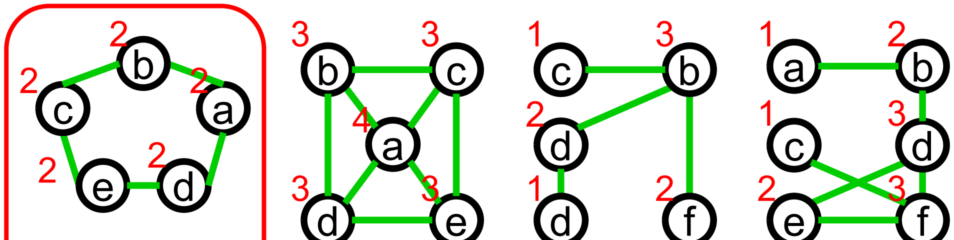


Ham. Path	YES	YES	NO	YES
Ham. Cycle	YES	YES	NO	NO
Vertex # (n)	5	5	5	6
All Deg ≥ n/2?	NO	YES	NO	NO

Hamilton Path

Ore's Theorem

- **Theorem**: If every pair of non-adjacent vertices u and v in a simple graph with n vertices and $n \geq 3$ has $deg(u)+deg(v) \geq n$, there is a Hamilton circuit



Ham. Path	YES	YES	NO	YES
Ham. Cycle	YES	YES	NO	NO
All deg(u)+deg(v) ≥ n	NO	YES	NO	NO

Dirac's and Ore's Theorem

- Be noted Dirac's and Ore's Theorem is a sufficient condition but not necessary one
  - A graph with a vertex degree < n/2 may have a Hamilton circuit
  - A graph with a pair of non-adjacent vertices deg(u)+deg(v) < n may have a Hamilton circuit

Hamilton Path

- Unfortunately, no good algorithm to find the Hamilton path or cycle
- Just "trial and error" (and good luck!)

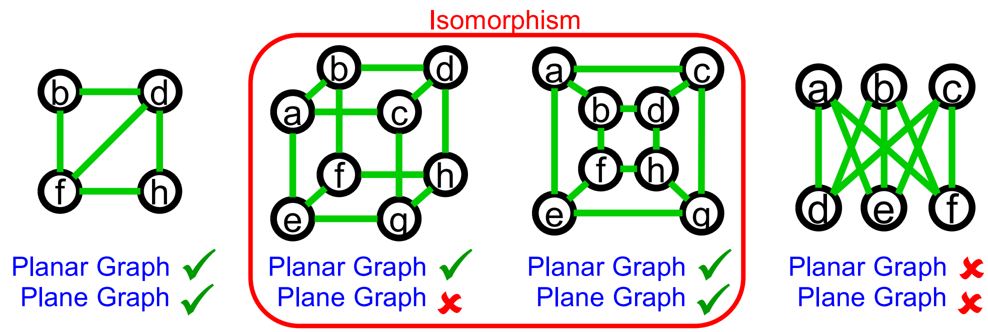
Euler Path VS Hamilton Path

- **Euler Path**
  - a path uses every edge exactly once
- **Euler Cycle**
  - Euler path which starts and stops at the same vertex

- **Hamilton Path**
  - a path uses every vertex exactly once
- **Hamilton Cycle**
  - Hamilton path which starts and stops at the same vertex

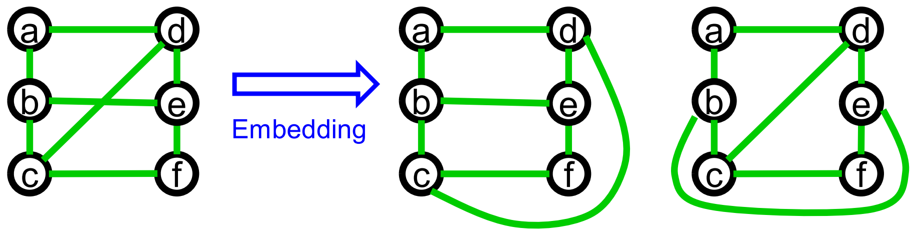
Planar Graph

- **Planar Graph** is a graph can be drawn in the plane without edges crossing
- A planar graph drawn in the plane without edges crossing is called **Plane Graph**
  - Plane graph is also called a planar representation



Planar Graph

- A graph that is drawn in the plane is also said to be embedded (or imbedded) in the plane
- A planar graph can generate different plane graphs
- Application: Circuit Layout Problems



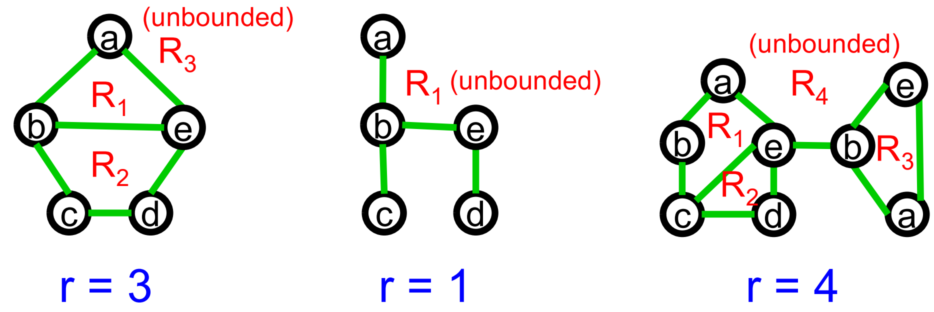
Not Plane Graph
Planar Graph

Plane Graph
Planar Graph

Plane Graph
Planar Graph

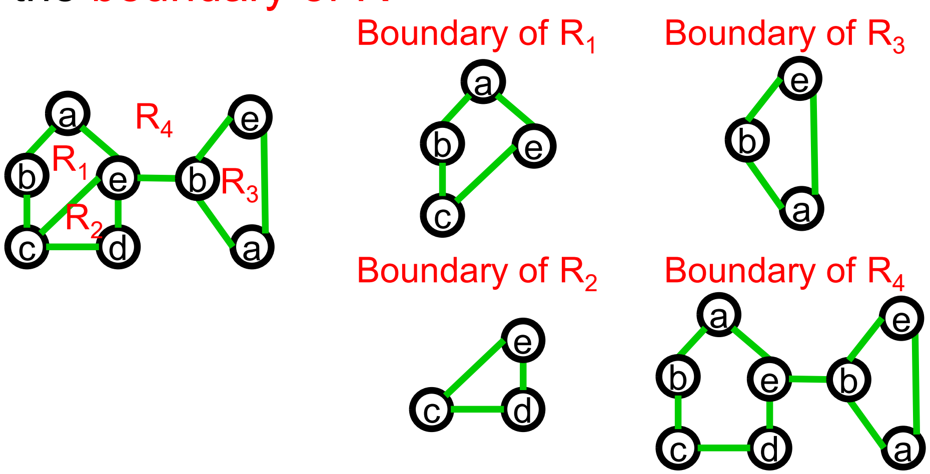
Planar Graph: Region

- A plane graph splits the plane into regions
  - Including the unbounded (exterior) region



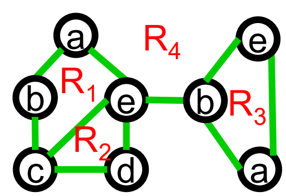
Planar Graph: Region

- The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R



Planar Graph: Region

- Observation on boundary
  - Cycle edge belongs to the boundary of two regions
  - Bridge is on the boundary of only one region (unbounded region)



a > b > c > e > a is a cycle

(a,b), (b,c), (a,e) belongs to R_1 and R_4

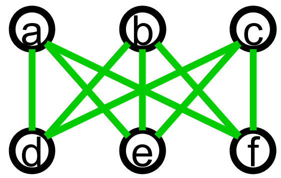
(c,e) belongs to R_1 and R_2

(e,b) is not a cycle, just a bridge

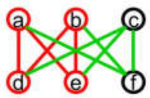
(e,b) belongs to R_4 only

Planar Graph

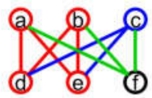
- Is $K_{3,3}$ a planar graph? Not planar



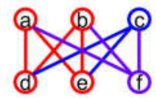
1) Focus on a, e, d, b

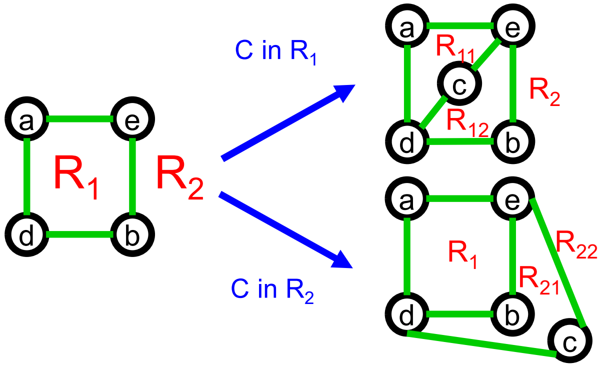


2) c connect to d, e, f



3) f connects to a, b, c





f in R_{11}: cross when connect to b
f in R_{12}: cross when connect to a
f in R_2: cross when connect to c

f in R_1: cross when connect to c
f in R_{21}: cross when connect to a
f in R_{22}: cross when connect to b

Planar Graph
Euler's Formula

- If G be a connected planar simple graph with e edges, v vertices, and r regions, then

$$r = e - v + 2$$

- MI is used in the proof

Planar Graph
Euler's Formula: Proof

$r = e - v + 2$

- For a connected planar graph G
  - Let a sequence of subgraphs $G_1, G_2, \ldots G_i, \ldots, G_e$ of G, and $G_e = G$,
    - $G_1 \subset G_2 \subset \ldots \subset G_e$
    - G_i contains i edges
    - G_n is obtained from G_{n-1} by arbitrarily adding an edge
  - Be noted that all G_i are planar (as subgraph of planar graph must be planar)

Planar Graph
Euler's Formula: Proof

$r = e - v + 2$

- For G_1,
  - $e_1 = 1$
  - $v_1 = 2$
  - $r_1 = 1$

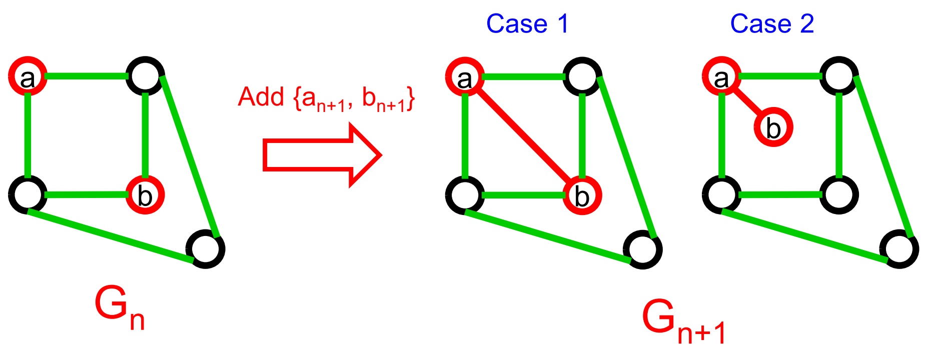
- Therefore, $r_1 = e_1 - v_1 + 2$
- Assume $r_n = e_n - v_n + 2$ is true

Planar Graph

Euler's Formula: Proof

$r = e - v + 2$

- Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}
  - Case 1: a_{n+1}, b_{n+1} are in G_n
  - Case 2: one of a_{n+1} and b_{n+1} is not in G_n



Planar Graph

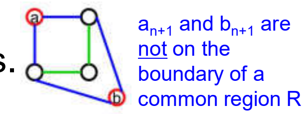
Euler's Formula: Proof

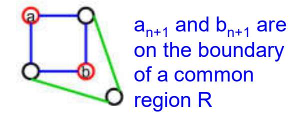
$r = e - v + 2$

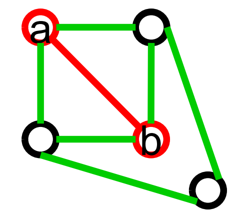
- Case 1: a_{n+1} and b_{n+1} are in G_n
  - $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$
  - If a_{n+1} and b_{n+1} are not on the boundary of a common region R, two edges cross. This violates G_{n+1} is planar
  - Therefore, a_{n+1} and b_{n+1} must be on the boundary of a common region R
  - The new edge splits R into two regions
    - $r_{n+1} = r_n + 1$
  - Given $r_n = e_n - v_n + 2$

$$(r_{n+1} - 1) = (e_{n+1} - 1) - (v_{n+1}) + 2$$

$$r_{n+1} = e_{n+1} - v_{n+1} + 2$$







Planar Graph

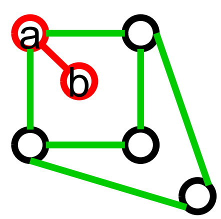
Euler's Formula: Proof

$r = e - v + 2$

- Case 2: one of a_{n+1} and b_{n+1} is not in G_n
  - $e_{n+1} = e_n + 1$
  - $v_{n+1} = v_n + 1$
  - No new region is generated, $r_{n+1} = r_n$
  - Given $r_n = e_n - v_n + 2$

$$(r_{n+1}) = (e_{n+1} - 1) - (v_{n+1} - 1) + 2$$

$$r_{n+1} = e_{n+1} - v_{n+1} + 2$$



Planar Graph

Euler's Formula: Example

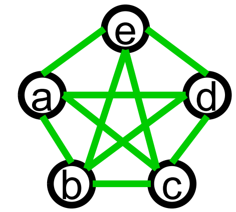
- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph slit the plane?
- v = 20
- Sum of degree = 20 x 3 = 60 = 2e
- e = 30
- r = e – v + 2 = 30 – 20 + 2 = 12

Euler's Formula: Corollary

- If a connected planar simple graph, then G has a vertex of degree not exceeding 5.
- If a connected planar simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$
- If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$

Euler's Formula: Example 1

- Show that K_5 is nonplanar

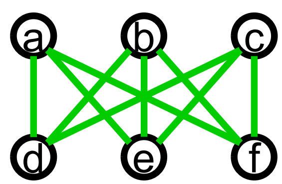


- K_5 has circuit of length three, 5 vertices and 10 edges
- As $e = 10$ and $3v - 6 = 9$, $e \leq 3v - 6$ is false
- Therefore, K_5 is nonplanar

If a connected planar simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$

Euler's Formula: Example 2

- Show that $K_{3,3}$ is nonplanar

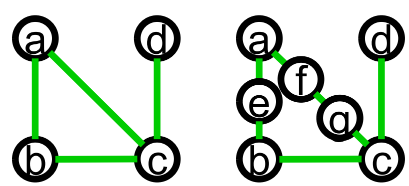


- $K_{3,3}$ has no circuit of length three, 6 vertices and 9 edges
- As $e = 9$ and $2v - 4 = 8$, $e \leq 2v - 4$ is false
- Therefore, $K_{3,3}$ is nonplanar

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$

Homeomorphic

- The graphs are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision
  - If a graph is planar, it will be any graph obtained by removing an edge {u,v} and adding a new vertex w with edges {u,w} and {w,v}



G H

Obtain G from H

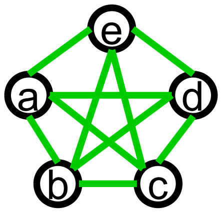
Remove {a, b}, Add {a, **e**}, {**e**, b}

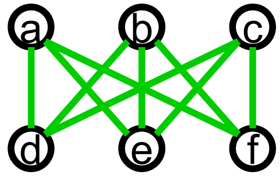
Remove {a, c}, Add {a, **f**}, {**f**, c}

Remove {f, c}, Add {f, **c**}, {**c**, g}

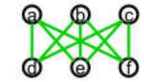
Kuratowski's Theorem

- A graph is not planar if it contains a non-planar subgraph
- **Kuratowski's Theorem**
  A graph is nonplanar iif it contains a subgraph homeomorphic to $K_{3,3}$ or K_5
- Proof is neglected





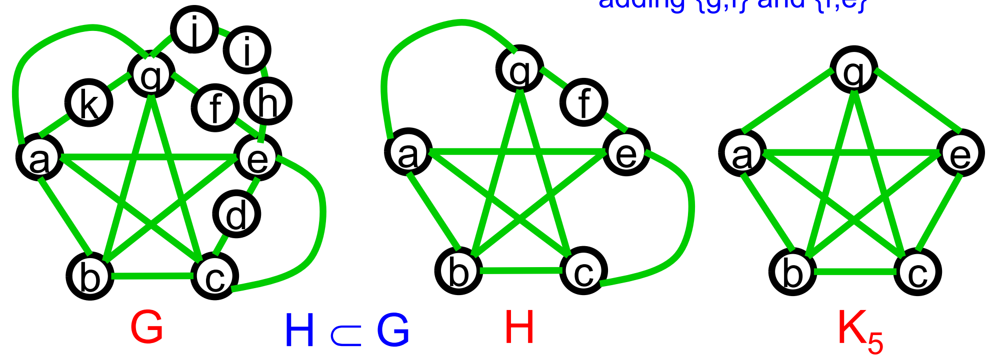
Example



- Determine whether the following graph is planar

H and K_5 are homeomorphic

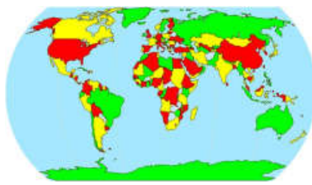
H can be obtained from K_5 by removing {g,e} and adding {g,f} and {f,e}



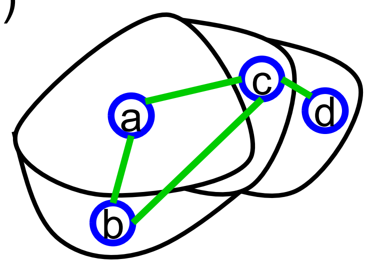
- As G contains a subgraph (H) homeomorphic to K_5, it is not planar

Coloring

- Two regions sharing a border are assigned different colors



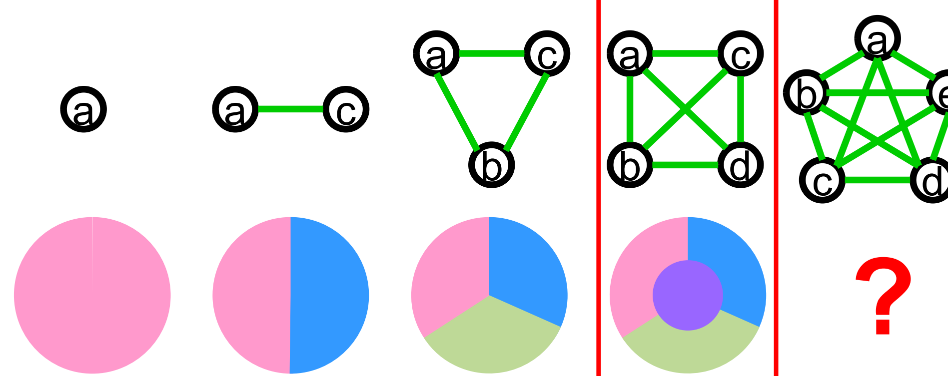
- Represent a map by a graph (called **Dual Graph**)
  - Vertex: Region
  - Edge: Constraint
    - the color cannot be the same for adjacent regions



Map Coloring

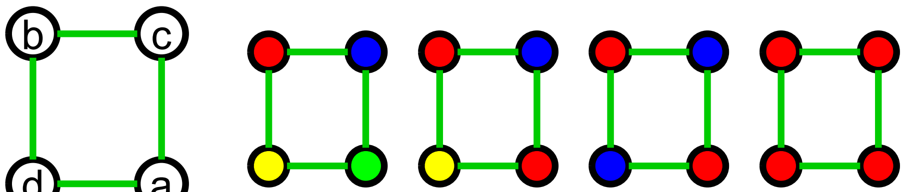
- What is the largest complete graph represented by a map?

Planar | Planar | Planar | Planar | nonPlanar



Coloring

- **Graph Coloring Problem**
  Given a graph, assign all the vertices with the minimum number of colors so that no two adjacent vertices gets the same color



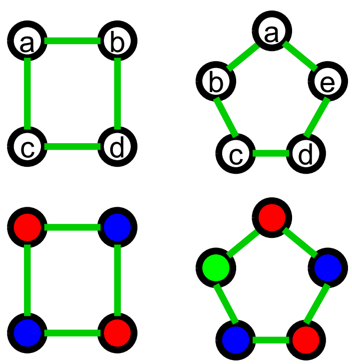
4 max   3   2   1 min

Coloring

- Chromatic number ($\chi(G)$)
  The smallest number of colors needed to produce a proper coloring of G

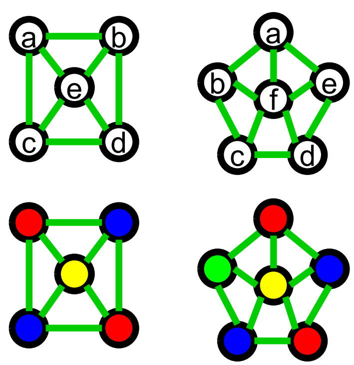
Coloring: Example

- Cycle Graph (C)



$\chi(C_{even})= 2$ $\chi(C_{odd})=3$

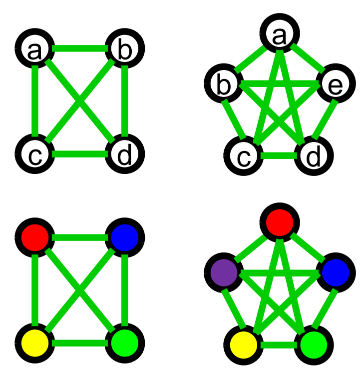
- Wheel Graph (W)



$\chi(W_{even})= 3$ $\chi(W_{odd})=4$

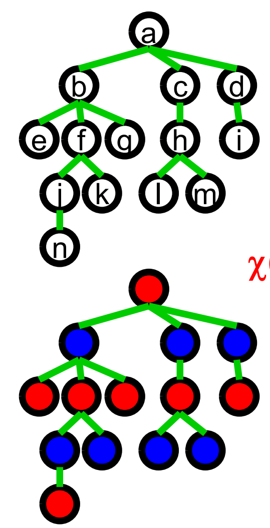
Coloring: Example

- Complete Graph (K)



$\chi(K_{even})= n$ $\chi(K_{odd})=n$

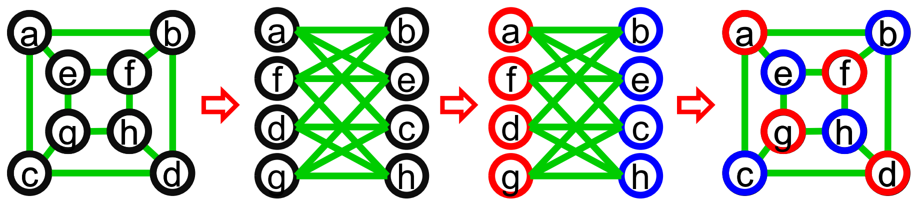
- Tree (T)



$\chi(T)= 2$

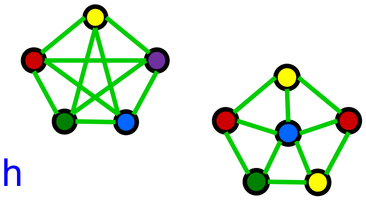
Coloring: Example

- Bipartite Graph
  - Recall… a graph is bipartite if all vertices can be partitioned into two partitions, so that any two adjacent vertices are in different partitions
  - Obviously, $\chi = 2$

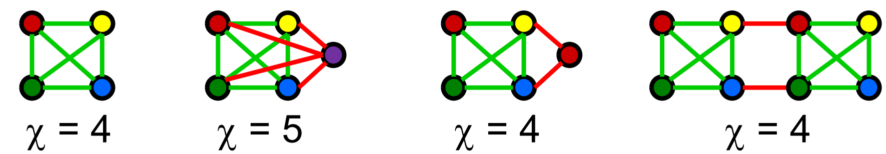


Coloring

- No formula for Chromatic number χ
- Discussion
  - Given a graph of size k
    - $\chi > k$: not possible
    - $\chi = k$: for a complete graph
    - $\chi < k$: other graphs except the complete one



  - Analyzing a subgraph of a graph may be helpful
    - If a subgraph is complete of size k, $\chi \geq k$

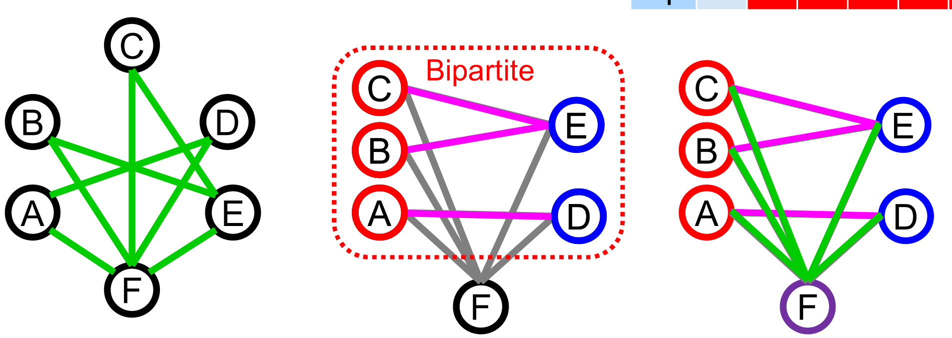


Coloring: Application 1

- A flight need a gate in an airport
- How many gates needed for this flight schedule? 3

	T1	T2	T3	T4	T5	T6
F_A		■				
F_B			■	■		
F_C					■	■
F_D	■	■				
F_E				■	■	
F_F		■	■	■	■	■

Vertex: Flight
Edge: Share the same time slot

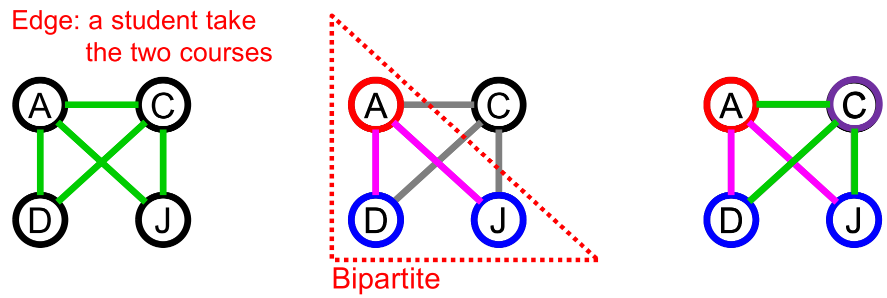


Coloring: Application 2

- Examination of subject conflicts if student takes both subjects
- How many different time slots? 3

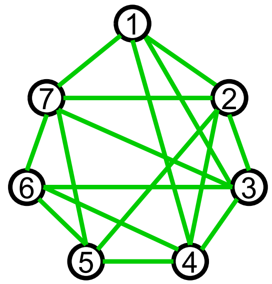
	S_1	S_2	S_3	S_4
AI		■		■
C++	■	■	■	■
DisMaths			■	■
Java	■	■		

Vertex: Course
Edge: a student take the two courses

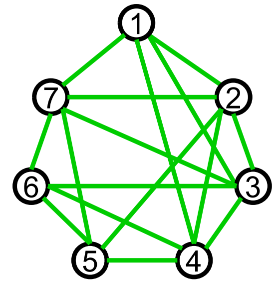


Coloring: Application 3

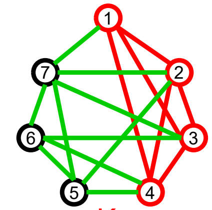
- Suppose an university offers seven courses. Students can take more than one course.
- Pairings of courses:
  - Course 1 : 2, 3, 4, 7
    - Course 1 has a student in common with courses 2, 3, 4, 7
  - Course 2 : 3, 4, 5, 7
  - Course 3 : 4, 6, 7
  - Course 4 : 5, 6
  - Course 5 : 6, 7
  - Course 6 : 7
- Find the fewest number of final exam slots that are needed to avoid any conflicts



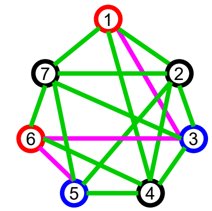
Coloring: Application 3



Answer is 4



K_4



Bipartite

7 is not connected to 4