Chapter 9: Graphs

9.5 Euler and Hamilton Paths

9.7 Planar Graphs

9.8 Graph Coloring

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Agenda

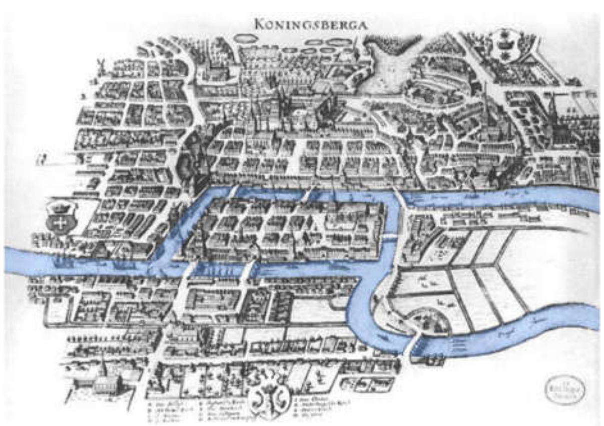
- Euler Path
- Hamilton Path
- Planar Graph
- Coloring

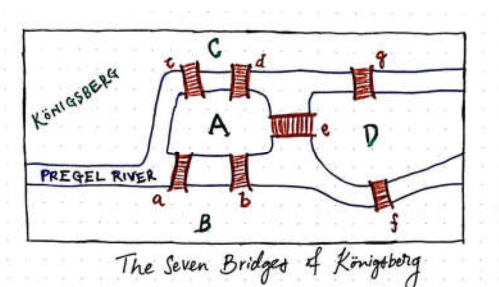
Euler and Hamiltonian Path

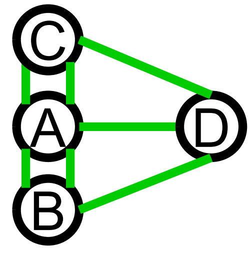
- **Euler Path**
  a path visits every edge exactly once
- **Hamiltonian Path**
  a path visits every vertex exactly once

Euler Path

- **Seven Bridges of Königsberg**
  - Königsberg is built on both banks of the Preger river
  - Now a city in Russia called Kaliningrad
- Is it possible to walk through the city that would cross each of bridges once





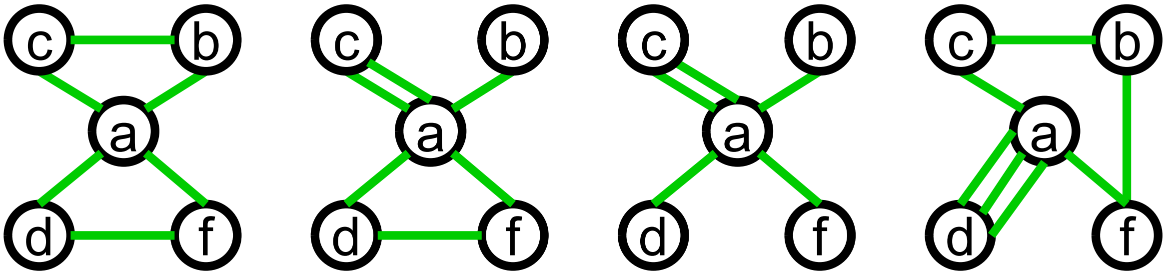


Euler Path

- **Leonhard Euler**, the Swiss mathematician, was also unable to find such a route
- Euler figured out how to show for certain that no such route existed

Euler Path

- Euler Path: a path visits every edge exactly once
- Euler Cycle: Euler path which starts and stops at the same vertex
- A connected graph G is called Eulerian if it contains an Euler path



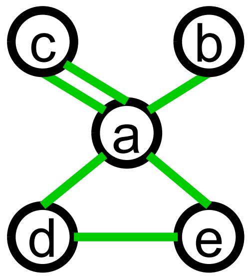
Euler Path	YES	YES	NO	YES
Euler Cycle	YES	NO	NO	NO
Eulerian	YES	YES	NO	YES

Euler Path

- Observation from an Euler path,

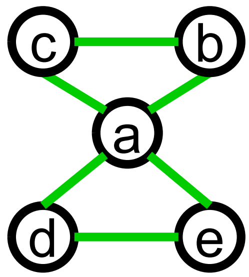
a > c > a > d > e > a > b

Euler Path



a > c > b > a > d > f > a

Euler Cycle



Starting / Intermediate	deg(a) = 5 (O)	deg(a) = 4 (E)	Starting / Intermediate / End
End	deg(b) = 1 (O)	deg(b) = 2 (E)	Intermediate
Intermediate	deg(c) = 2 (E)	deg(c) = 2 (E)	Intermediate
Intermediate	deg(d) = 2 (E)	deg(d) = 2 (E)	Intermediate
Intermediate	deg(e) = 2 (E)	deg(e) = 2 (E)	Intermediate

Euler Path

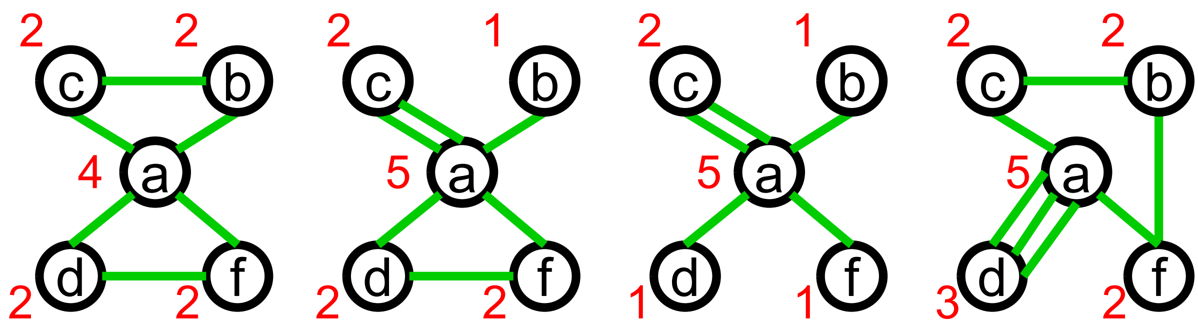
- Observation from an Euler path,
  - Intermediate vertex
    - Degree must be even
      (Entrance and exist connection)
  - Starting and end vertices
    - If the same (cycle),
      degree are even
    - If different (non-cycle),
      degrees are odd (in or out)

Euler Path

- **Theorem 1**
  A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree

- **Theorem 2**
  A connected multigraph has Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree

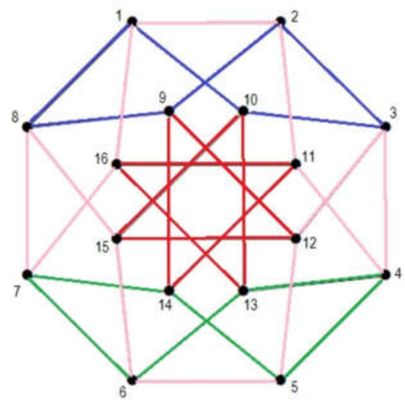
Euler Path



Euler Path **YES**	Euler Path **YES**	Euler Path **NO**	Euler Path **YES**
Euler Cycle **YES**	Euler Cycle **NO**	Euler Cycle **NO**	Euler Cycle **NO**
Eulerian **YES**	Eulerian **YES**	Eulerian **NO**	Eulerian **YES**
Odd Degree **0**	Odd Degree **2**	Odd Degree **4**	Odd Degree **2**
Even Degree **5**	Even Degree **3**	Even Degree **1**	Even Degree **3**
	Starting/End Vertices must be a and c		Starting/End Vertices must be a and d

Euler Path

- How to identify an Euler Path / Cycle?
  - Euler Path
    Fleury's Algorithm
  - Euler Cycle
    Hierholzer's Algorithm

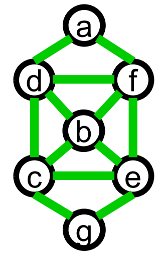


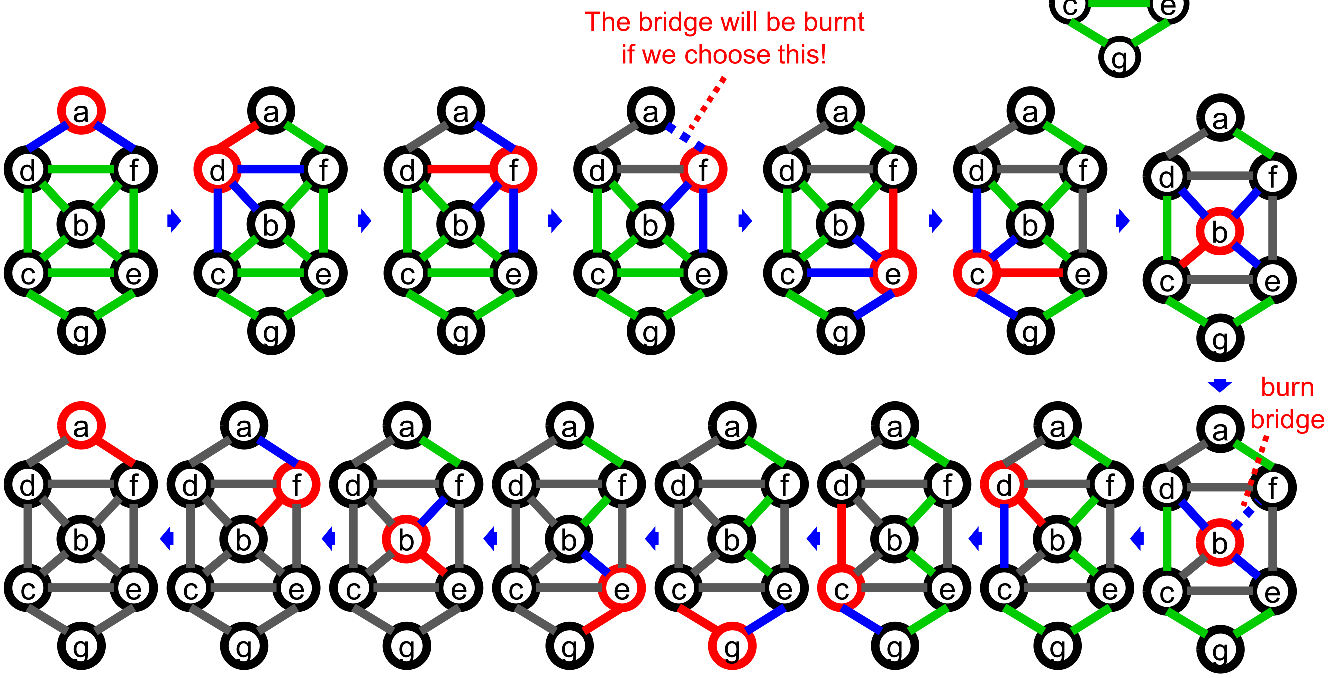
Euler Path

Fleury's Algorithm

- Identify Euler Path
  1. If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them
  2. Follow edges one at a time. If you have a choice between a bridge and a non-bridge, always choose the non-bridge
  3. Stop when you run out of edges
- "Don't burn bridges" so that we can come back to a vertex and traverse remaining edges

Fleury's Algorithm



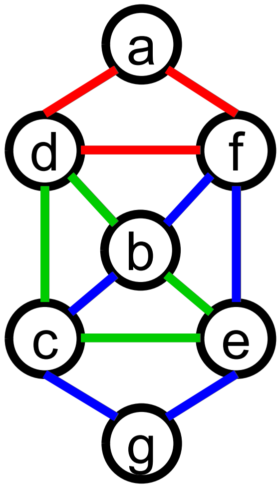


Hierholzer's Algorithm

- Identify Euler Cycle
  1. Select a node v as a starting node
  2. Form a cycle using non-traveled edges and end at v (remove the visited edges)
  3. While all edges have been traversed, stop
     a) Find a node u on the previous cycles that's connected to a non-traveled edge
     b) Form a cycle using non-traveled edges and end at u (remove the visited edges)
     c) Merge both tours at the node u

Euler Path
Hierholzer

1. Select a node v as a starting node
2. Form a cycle using non-traveled edges and end at v (remove the visited edges)
3. While all edges have been traversed, stop
   a) Find a node v on the previous cycles that's connected to a non-traveled edge
   b) Form a cycle using non-traveled edges and end at v (remove the visited edges)
   c) Merge both tours at the node v



a > d > f > a

d > b > e > c > d

f > b > c > g > e > f

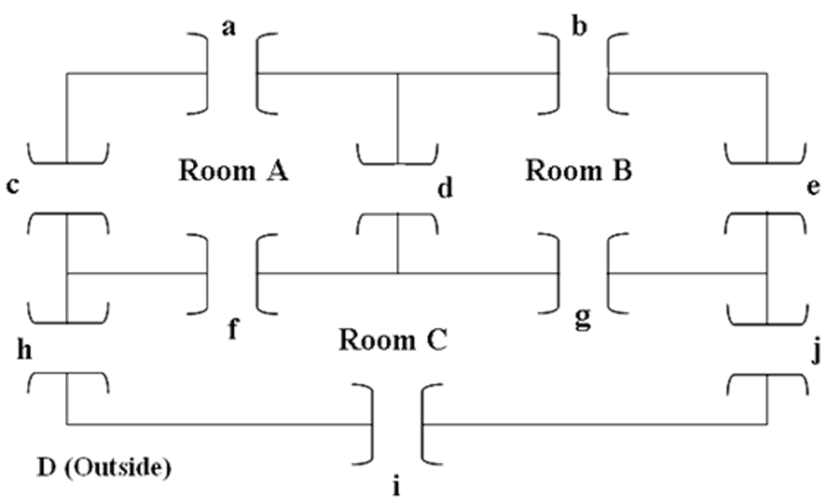
a > d > f > a

d > b > e > c > d   f > b > c > g > e > f

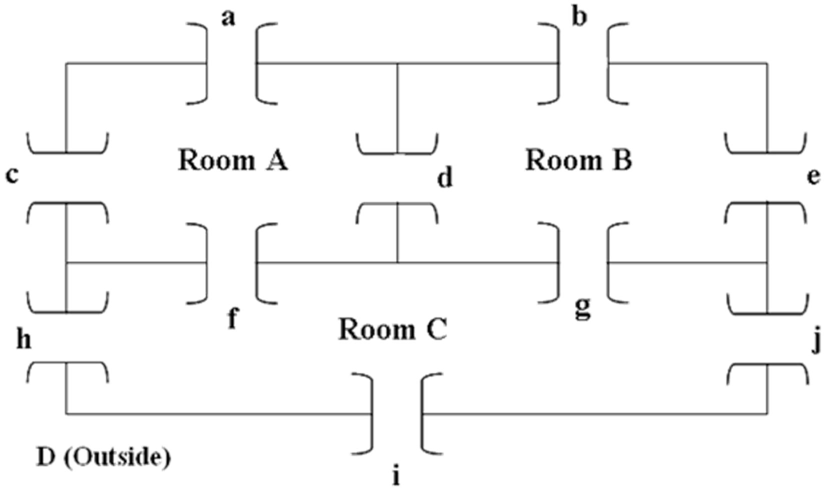
a > d > b > e > c > d > f > b > c > g > e > f > a

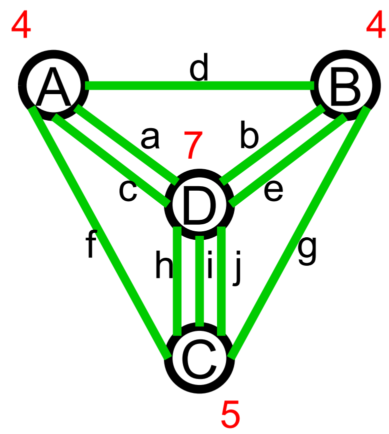
Euler Path
Example 1

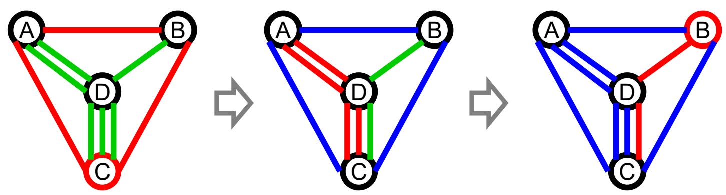
- Is it possible to begin in a room or the outside and take a walk that goes through each door exactly once? If yes, how?



Example 1

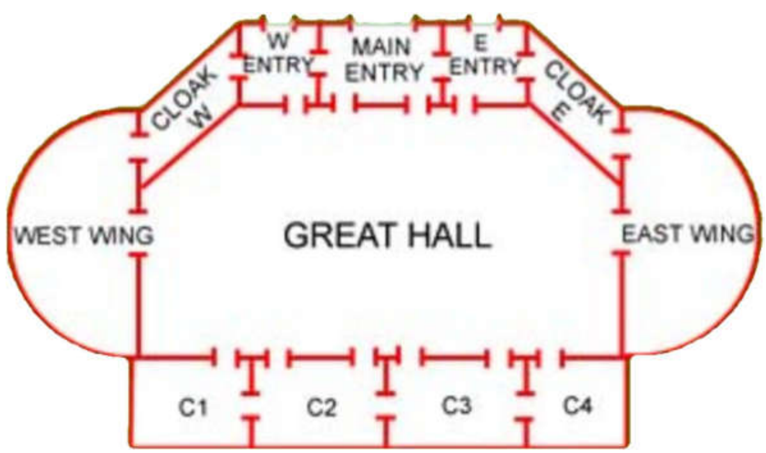




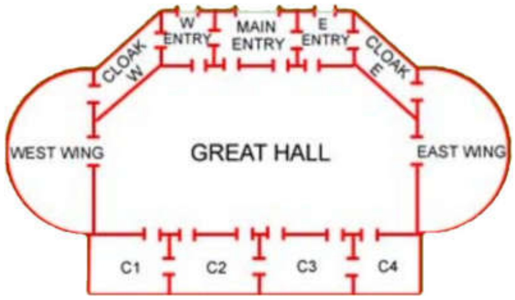


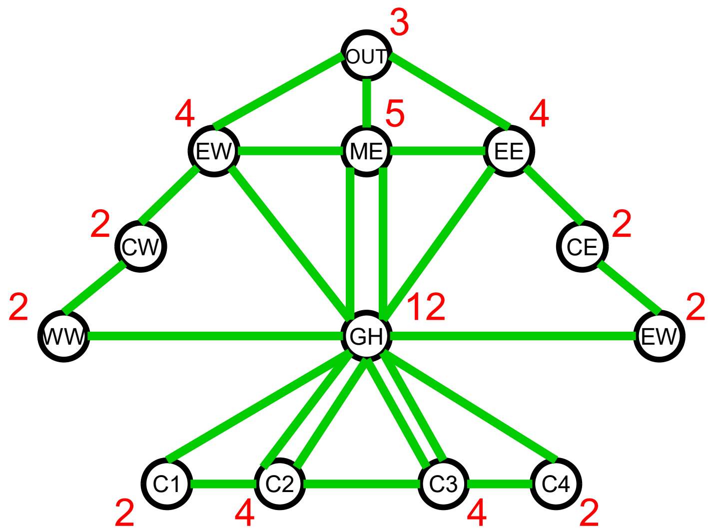
Example 2

- Is it possible to walk through and around this building passing through each and every doorway exactly once?

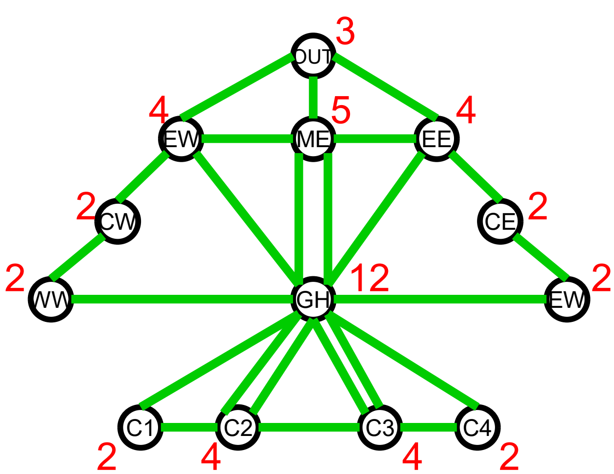


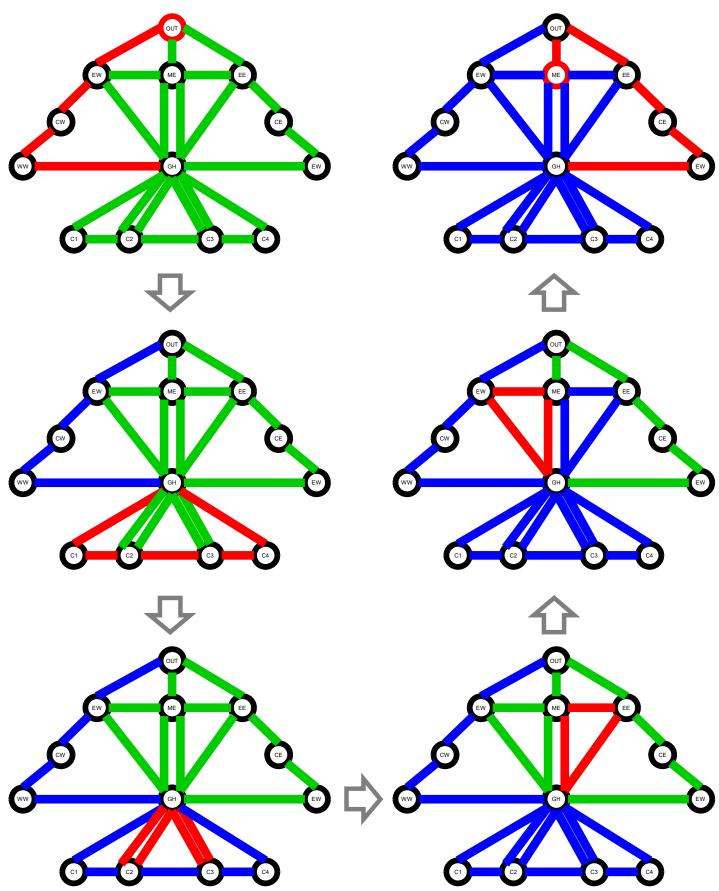
Euler Path
Example 2





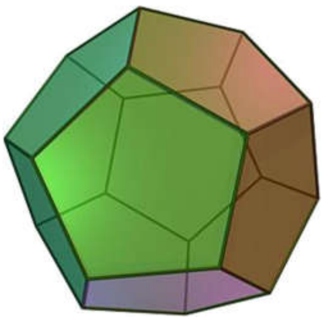
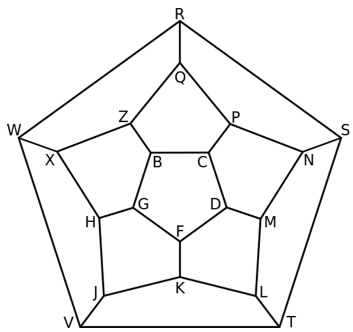
Euler Path
Example 2





Hamiltonian Path

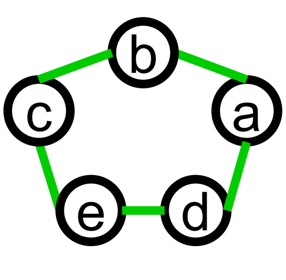
- Icosian game
  - Invented by an Irishman named Sir William Rowan Hamilton (1805-1865)
  - Is there a cycle in the dodecahedron puzzle that passes through each vertex exactly once?

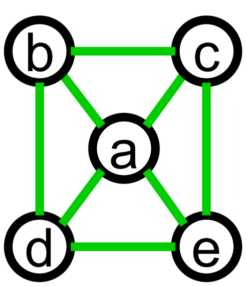
Dodecahedron puzzle

Hamilton Path

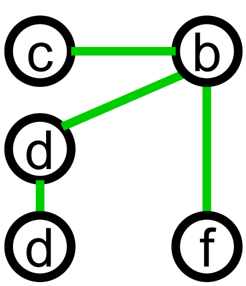
- Hamilton Path: a path visits every vertex exactly once
- Hamilton Cycle: Hamilton path which starts and stops at the same vertex
- Self-loop and multiple edges can be ignored



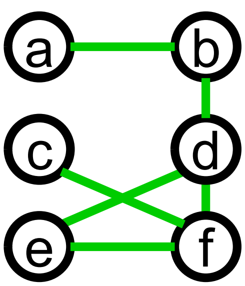
Ham. Path YES
Ham. Cycle YES



Ham. Path YES
Ham. Cycle YES



Ham. Path NO
Ham. Cycle NO

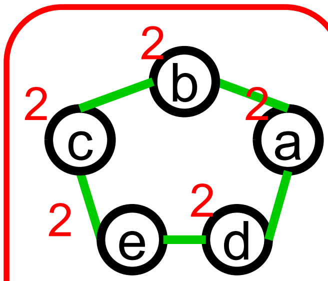


Ham. Path YES
Ham. Cycle NO

Hamilton Path

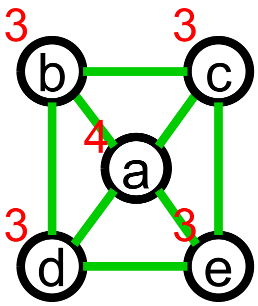
Dirac's Theorem

- **Theorem**: If each vertex of a simple graph with n vertices and n ≥ 3 has degree ≥ n/2, there is Hamilton circuit



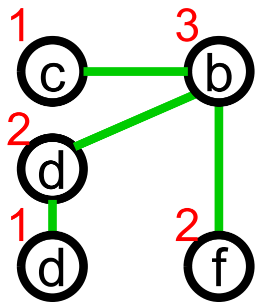
Ham. Path **YES**
Ham. Cycle **YES**

Vertex # (n) **5**
All Deg ≥ n/2? **NO**



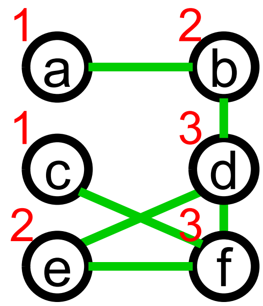
Ham. Path **YES**
Ham. Cycle **YES**

Vertex # (n) **5**
All Deg ≥ n/2? **YES**



Ham. Path **NO**
Ham. Cycle **NO**

Vertex # (n) **5**
All Deg ≥ n/2? **NO**



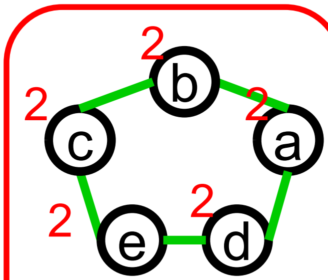
Ham. Path **YES**
Ham. Cycle **NO**

Vertex # (n) **6**
All Deg ≥ n/2? **NO**

Hamilton Path

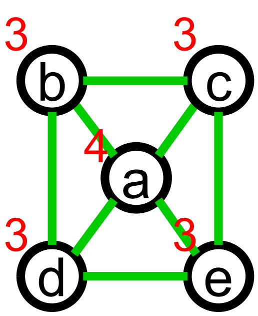
Ore's Theorem

- **Theorem**: If every pair of non-adjacent vertices u and v in a simple graph with n vertices and n ≥ 3 has deg(u)+deg(v) ≥ n, there is a Hamilton circuit



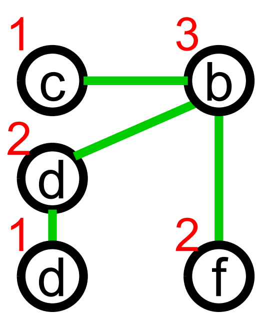
Ham. Path **YES**
Ham. Cycle **YES**

All deg(u)+deg(v) ≥ n
NO



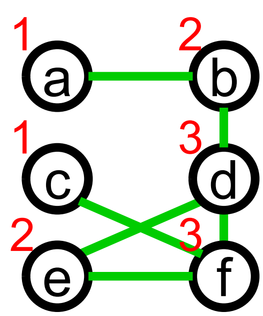
Ham. Path **YES**
Ham. Cycle **YES**

All deg(u)+deg(v) ≥ n
YES



Ham. Path **NO**
Ham. Cycle **NO**

All deg(u)+deg(v) ≥ n
NO



Ham. Path **YES**
Ham. Cycle **NO**

All deg(u)+deg(v) ≥ n
NO

Dirac's and Ore's Theorem

- Be noted Dirac's and Ore's Theorem is a sufficient condition but not necessary one
  - A graph with a vertex degree < n/2 may have a Hamilton circuit
  - A graph with a pair of non-adjacent vertices deg(u)+deg(v) < n may have a Hamilton circuit

Hamilton Path

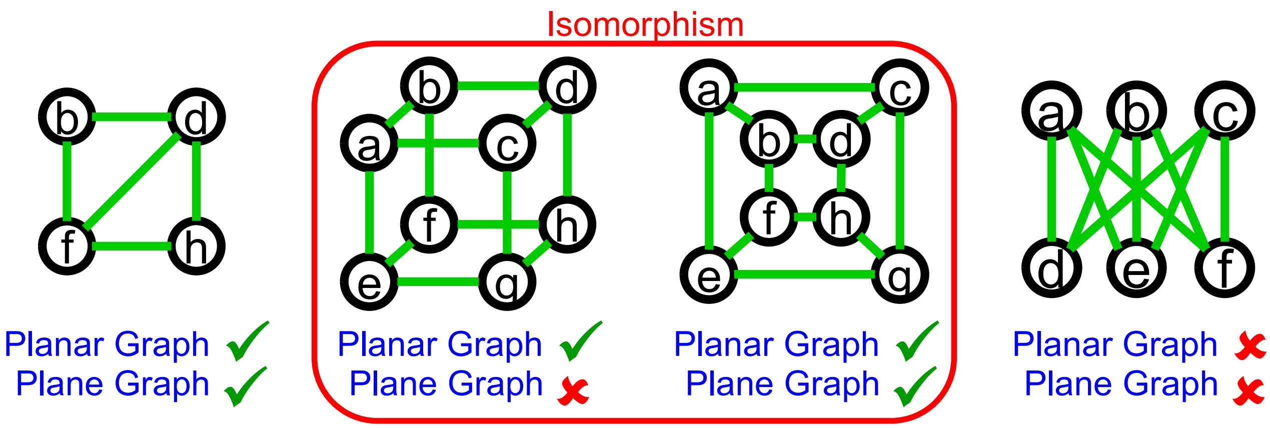
- Unfortunately, no good algorithm to find the Hamilton path or cycle
- Just "trial and error" (and good luck!)

Euler Path VS Hamilton Path

- **Euler Path**
  - a path uses every edge exactly once
- **Euler Cycle**
  - Euler path which starts and stops at the same vertex
- **Hamilton Path**
  - a path uses every vertex exactly once
- **Hamilton Cycle**
  - Hamilton path which starts and stops at the same vertex

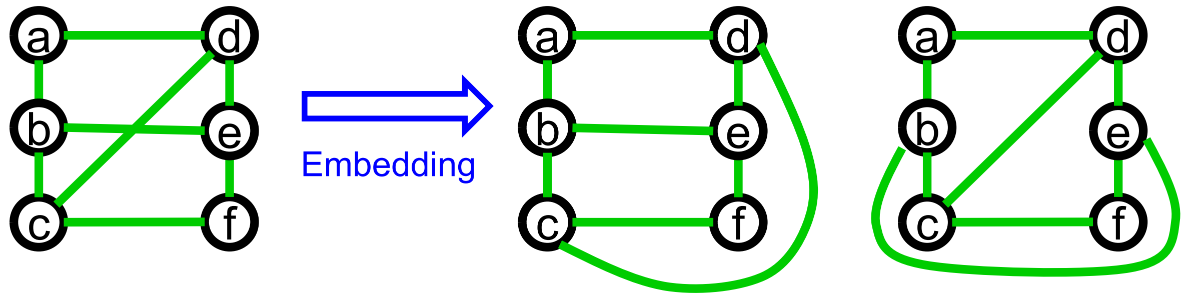
Planar Graph

- **Planar Graph** is a graph can be drawn in the plane without edges crossing
- A planar graph drawn in the plane without edges crossing is called **Plane Graph**
  - Plane graph is also called a planar representation



Planar Graph

- A graph that is drawn in the plane is also said to be embedded (or imbedded) in the plane
- A planar graph can generate different plane graphs
- Application: Circuit Layout Problems



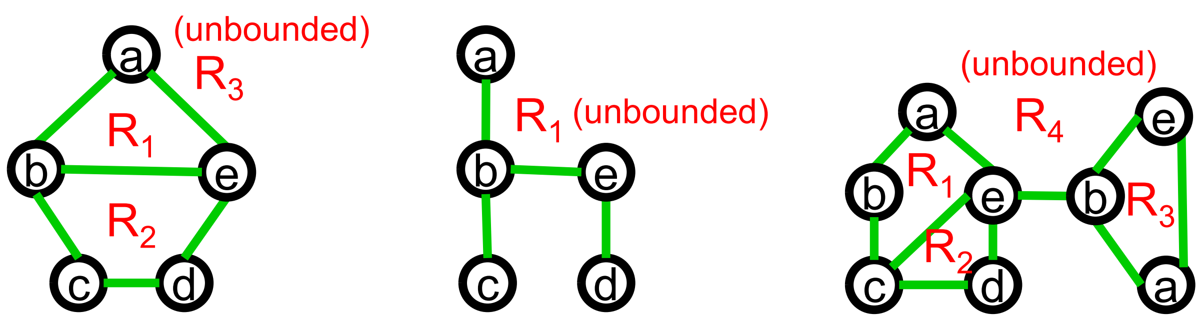
Not Plane Graph
Planar Graph

Plane Graph
Planar Graph

Plane Graph
Planar Graph

Planar Graph: Region

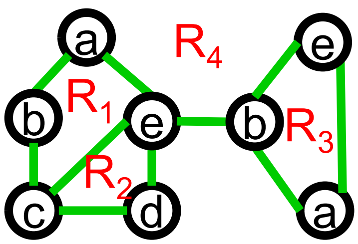
- A plane graph splits the plane into regions
  - Including the unbounded (exterior) region



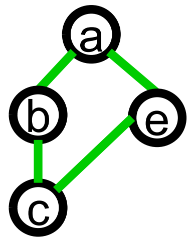
r = 3   r = 1   r = 4

Planar Graph: Region

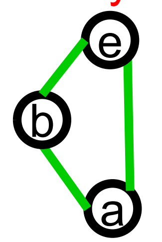
- The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R



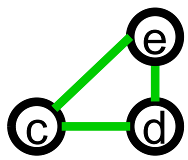
Boundary of R_1



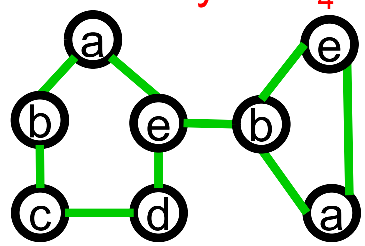
Boundary of R_3



Boundary of R_2

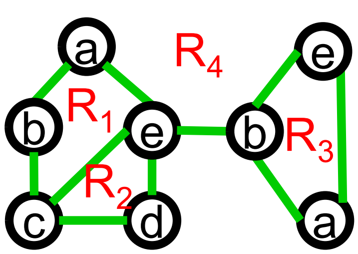


Boundary of R_4



Planar Graph: Region

- Observation on boundary
  - Cycle edge belongs to the boundary of two regions
  - Bridge is on the boundary of only one region (unbounded region)



a > b > c > e > a is a cycle

(a,b), (b,c), (a,e) belongs to R_1 and R_4

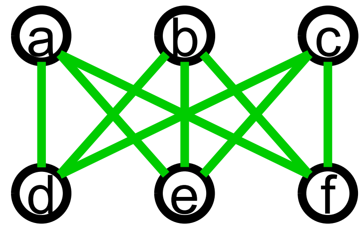
(c,e) belongs to R_1 and R_2

(e,b) is not a cycle, just a bridge

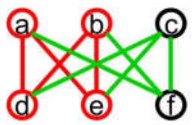
(e,b) belongs to R_4 only

Planar Graph

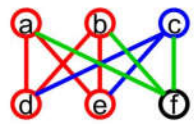
- Is $K_{3,3}$ a planar graph? Not planar



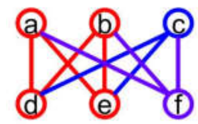
1) Focus on a, e, d, b

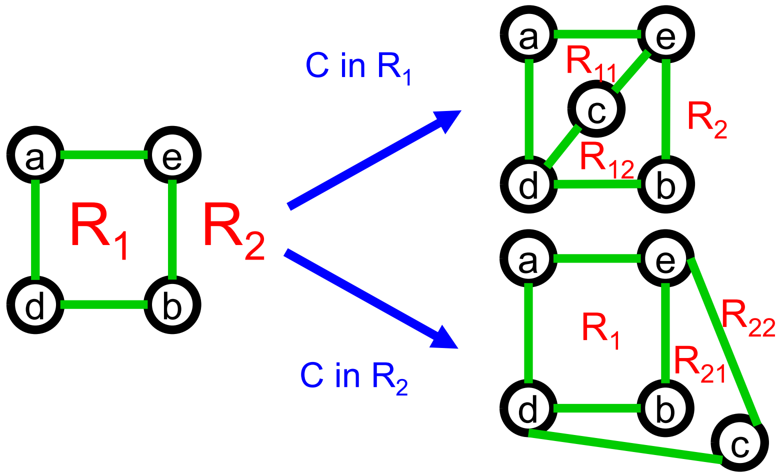


2) c connect to d, e, f



3) f connects to a, b, c





f in R_{11}: cross when connect to b
f in R_{12}: cross when connect to a
f in R_2: cross when connect to c

f in R_1: cross when connect to c
f in R_{21}: cross when connect to a
f in R_{22}: cross when connect to b

Planar Graph

Euler's Formula

- If G be a connected planar simple graph with e edges, v vertices, and r regions, then

$$r = e - v + 2$$

- MI is used in the proof

Euler's Formula: Proof

$r = e - v + 2$

- For a connected planar graph G
  - Let a sequence of subgraphs $G_1, G_2, \ldots G_i, \ldots, G_e$ of G, and $G_e = G$,
    - $G_1 \subset G_2 \subset \ldots \subset G_e$
    - G_i contains i edges
    - G_n is obtained from G_{n-1} by arbitrarily adding an edge
  - Be noted that all G_i are planar (as subgraph of planar graph must be planar)

Euler's Formula: Proof

$r = e - v + 2$

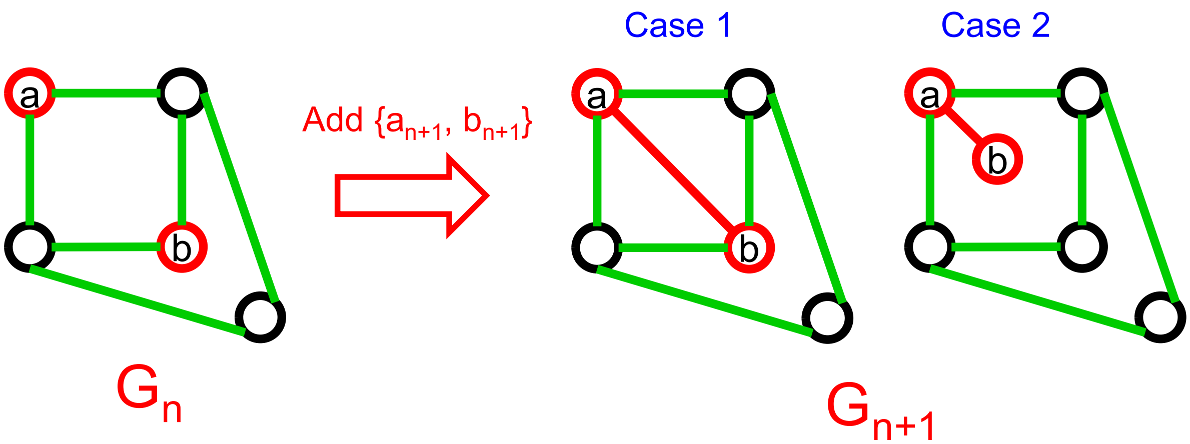
- For G_1,
  - $e_1 = 1$
  - $v_1 = 2$
  - $r_1 = 1$

- Therefore, $r_1 = e_1 - v_1 + 2$
- Assume $r_n = e_n - v_n + 2$ is true

Euler's Formula: Proof

$r = e - v + 2$

- Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}
  - Case 1: a_{n+1}, b_{n+1} are in G_n
  - Case 2: one of a_{n+1} and b_{n+1} is not in G_n



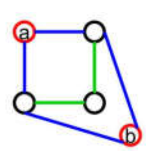
Euler's Formula: Proof

$r = e - v + 2$

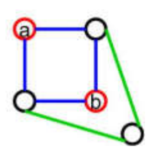
- Case 1: a_{n+1} and b_{n+1} are in G_n
  - $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$
  - If a_{n+1} and b_{n+1} are not on the boundary of a common region R, two edges cross. This violates G_{n+1} is planar
  - Therefore, a_{n+1} and b_{n+1} must be on the boundary of a common region R
  - The new edge splits R into two regions
    - $r_{n+1} = r_n + 1$
  - Given $r_n = e_n - v_n + 2$

$$(r_{n+1} - 1) = (e_{n+1} - 1) - (v_{n+1}) + 2$$

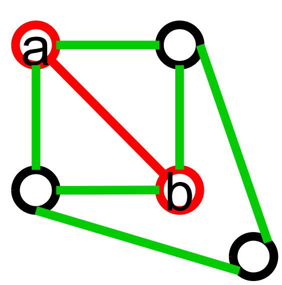
$$r_{n+1} = e_{n+1} - v_{n+1} + 2$$



a_{n+1} and b_{n+1} are not on the boundary of a common region R



a_{n+1} and b_{n+1} are on the boundary of a common region R



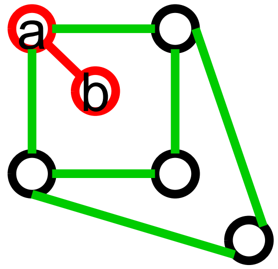
Euler's Formula: Proof

$r = e – v + 2$

- Case 2: one of a_{n+1} and b_{n+1} is not in G_n
  - $e_{n+1} = e_n + 1$
  - $v_{n+1} = v_n + 1$
  - No new region is generated, $r_{n+1} = r_n$
  - Given $r_n = e_n – v_n + 2$

$$(r_{n+1}) = (e_{n+1} – 1) – (v_{n+1} – 1) + 2$$

$$r_{n+1} = e_{n+1} – v_{n+1} + 2$$



Euler's Formula: Example

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph slit the plane?

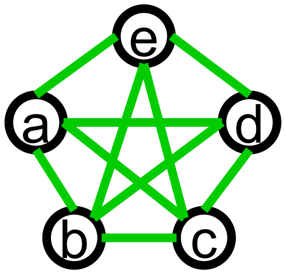
- v = 20
- Sum of degree = 20 x 3 = 60 = 2e
- e = 30
- r = e – v + 2 = 30 – 20 + 2 = 12

Euler's Formula: Corollary

- If a connected planar simple graph, then G has a vertex of degree not exceeding 5.
- If a connected planar simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$
- If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$

Euler's Formula: Example 1

- Show that K_5 is nonplanar



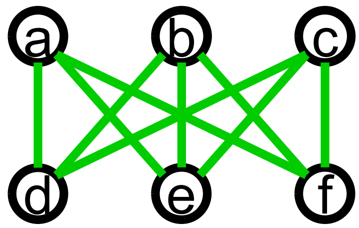
- K_5 has circuit of length three, 5 vertices and 10 edges
- As $e = 10$ and $3v - 6 = 9$, $e \leq 3v - 6$ is false
- Therefore, K_5 is nonplanar

If a connected planar simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$

Planar Graph

Euler's Formula: Example 2

- Show that $K_{3,3}$ is nonplanar



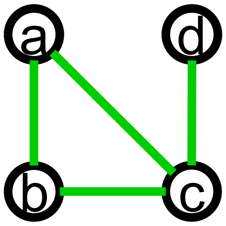
- $K_{3,3}$ has no circuit of length three, 6 vertices and 9 edges
- As $e = 9$ and $2v – 4 = 8$, $e \le 2v – 4$ is false
- Therefore, $K_{3,3}$ is nonplanar

If a connected planar simple graph has e edges and v vertices with $v \ge 3$ and no circuits of length three, then $e \le 2v – 4$

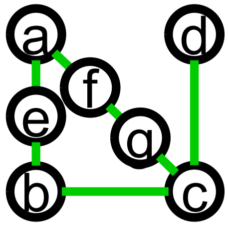
Planar Graph

Homeomorphic

- The graphs are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision
  - If a graph is planar, it will be any graph obtained by removing an edge {u,v} and adding a new vertex w with edges {u,w} and {w,v}



G



H

Obtain G from H

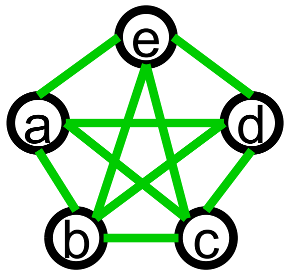
Remove {a, b}, Add {a, **e**}, {**e**, b}

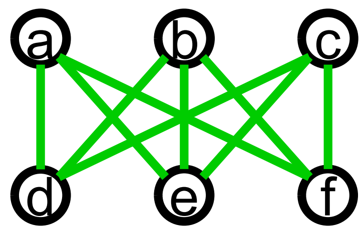
Remove {a, c}, Add {a, **f**}, {**f**, c}

Remove {f, c}, Add {f, **c**}, {**c**, g}

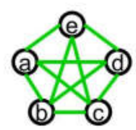
Kuratowski's Theorem

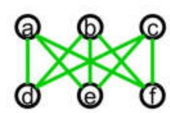
- A graph is not planar if it contains a non-planar subgraph
- **Kuratowski's Theorem**
  A graph is nonplanar iif it contains a subgraph homeomorphic to $K_{3,3}$ or K_5
- Proof is neglected





Example

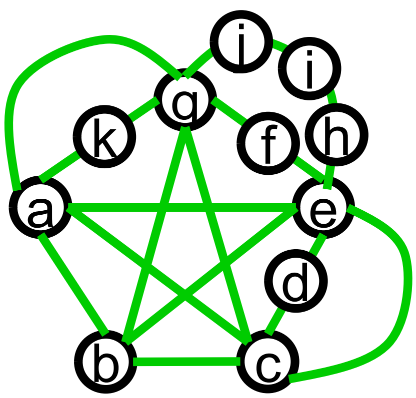




- Determine whether the following graph is planar

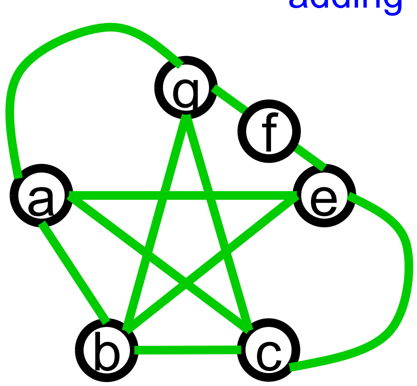
H and K_5 are homeomorphic

H can be obtained from K_5 by removing {g,e} and adding {g,f} and {f,e}

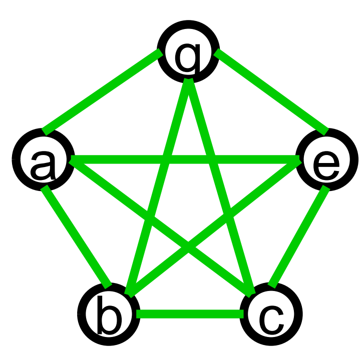


G

$H \subset G$



H

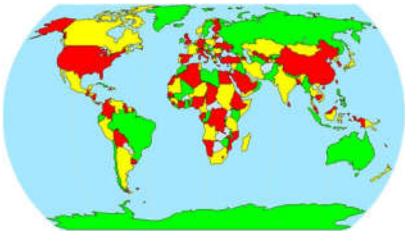


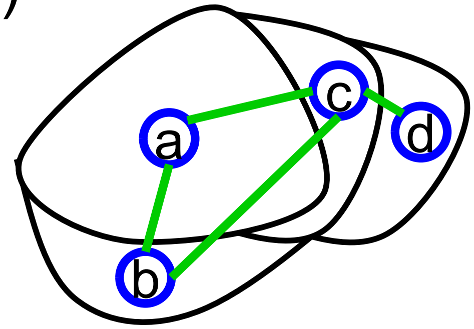
K_5

  - As G contains a subgraph (H) homeomorphic to K_5, it is not planar

Coloring

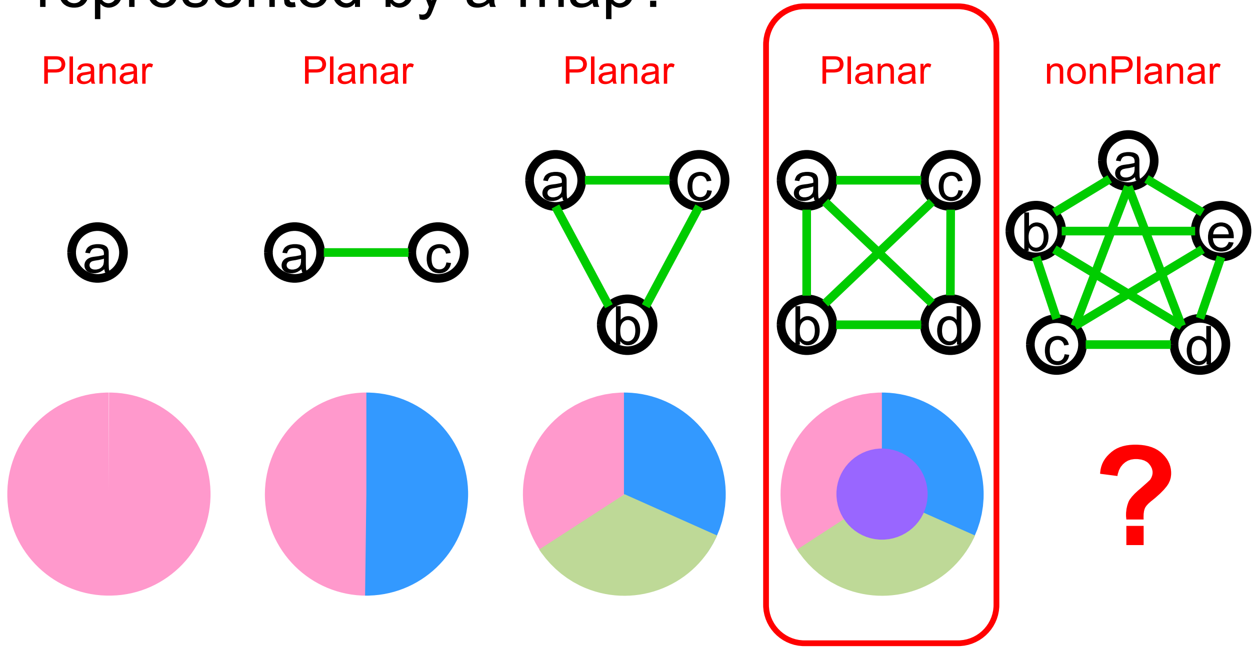
- Two regions sharing a border are assigned different colors
- Represent a map by a graph (called **Dual Graph**)
  - Vertex: Region
  - Edge: Constraint
    - the color cannot be the same for adjacent regions





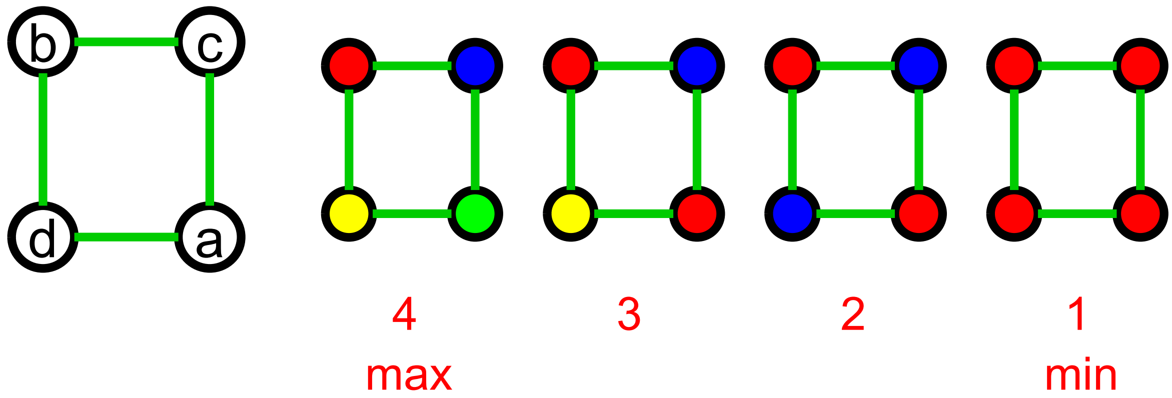
Map Coloring

- What is the largest complete graph represented by a map?



Coloring

- **Graph Coloring Problem**
  Given a graph, assign all the vertices with the minimum number of colors so that no two adjacent vertices gets the same color

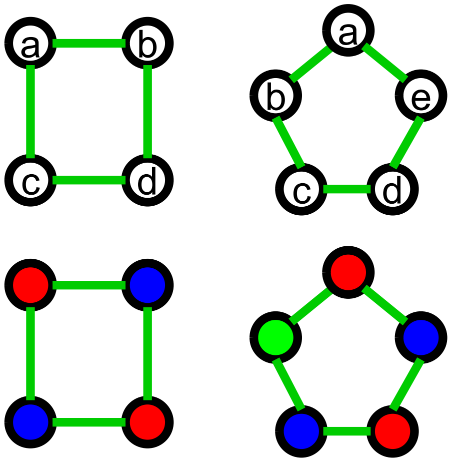


Coloring

- Chromatic number ($\chi(G)$)
  The smallest number of colors needed to produce a proper coloring of G

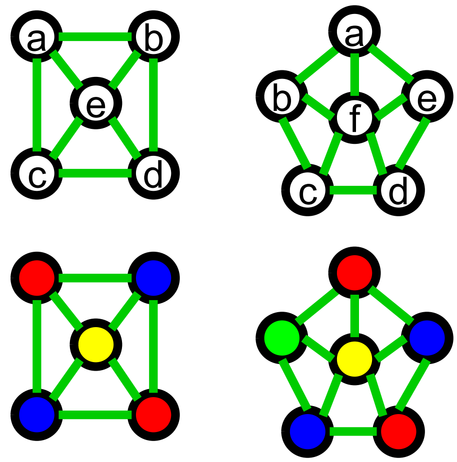
Coloring: Example

- Cycle Graph (C)



$\chi(C_{even})=2$ $\chi(C_{odd})=3$

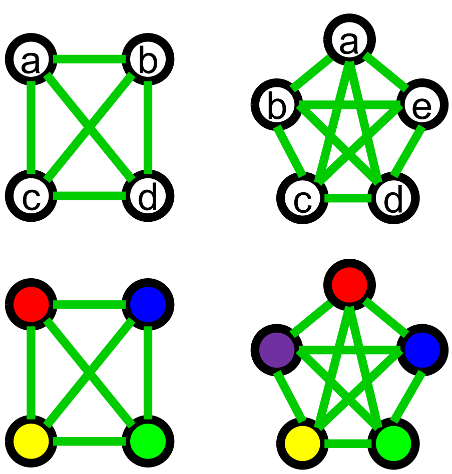
- Wheel Graph (W)



$\chi(W_{even})=3$ $\chi(W_{odd})=4$

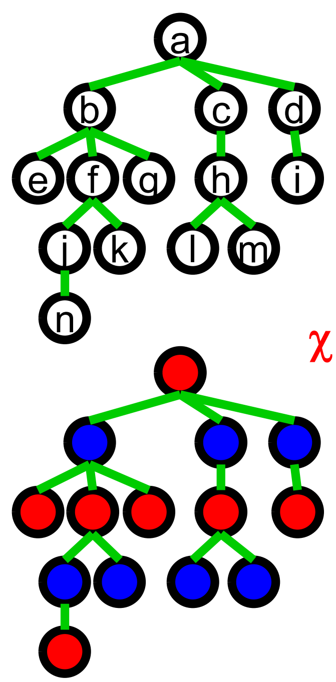
Coloring: Example

- Complete Graph (K)



$\chi(K_{even})=n$ $\chi(K_{odd})=n$

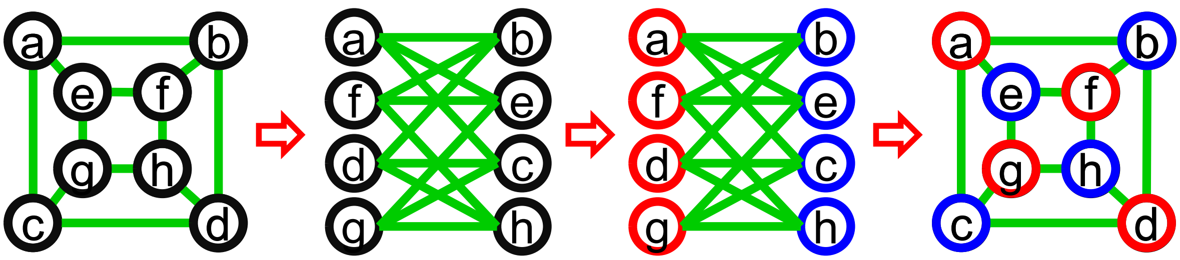
- Tree (T)



$\chi(T)=2$

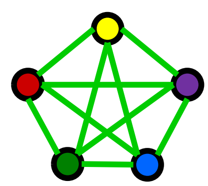
Coloring: Example

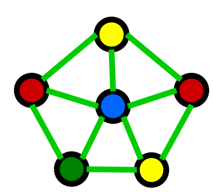
- Bipartite Graph
  - Recall… a graph is bipartite if all vertices can be partitioned into two partitions, so that any two adjacent vertices are in different partitions
  - Obviously, χ = 2



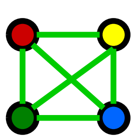
Coloring

- No formula for Chromatic number χ
- Discussion
  - Given a graph of size k
    - χ > k: not possible
    - χ = k: for a complete graph
    - χ < k: other graphs except the complete one

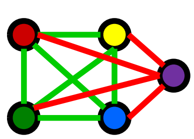




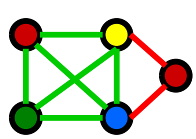
  - Analyzing a subgraph of a graph may be helpful
    - If a subgraph is complete of size k, $\chi \geq$ k



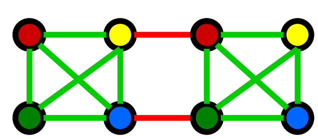
χ = 4



χ = 5



χ = 4



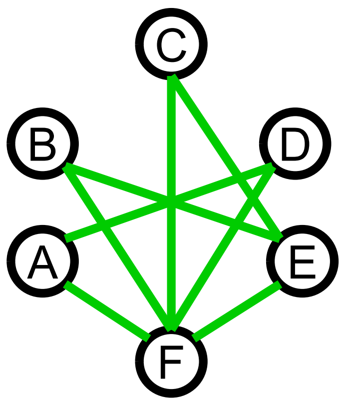
χ = 4

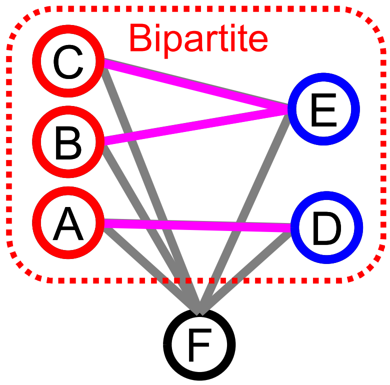
Coloring: Application 1

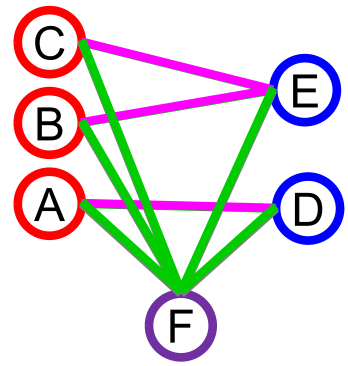
- A flight need a gate in an airport
- How many gates needed for this flight schedule? 3

	T1	T2	T3	T4	T5	T6
F_A						
F_B						
F_C						
F_D						
F_E						
F_F						

Vertex: Flight
Edge: Share the same time slot





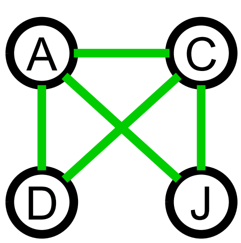


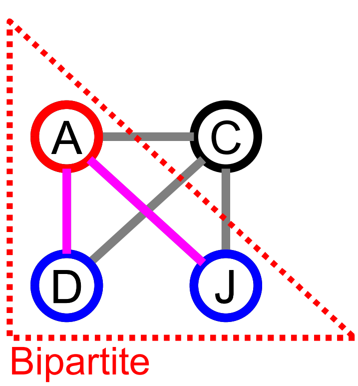
Coloring: Application 2

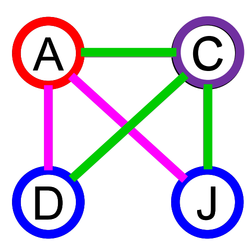
- Examination of subject conflicts if student takes both subjects
- How many different time slots? 3

	S_1	S_2	S_3	S_4
AI				
C++				
DisMaths				
Java				

Vertex: Course
Edge: a student take the two courses

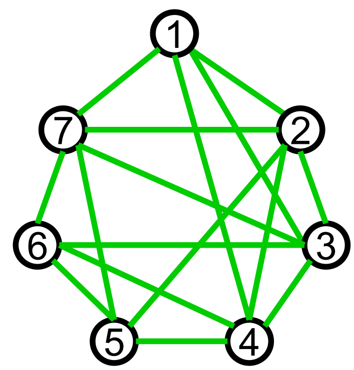






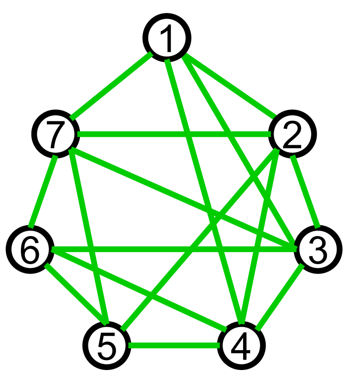
Coloring: Application 3

- Suppose an university offers seven courses. Students can take more than one course.
- Pairings of courses:
  - Course 1 : 2, 3, 4, 7
    - Course 1 has a student in common with courses 2, 3, 4, 7
  - Course 2 : 3, 4, 5, 7
  - Course 3 : 4, 6, 7
  - Course 4 : 5, 6
  - Course 5 : 6, 7
  - Course 6 : 7

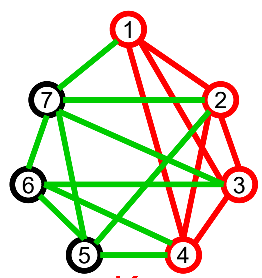


- Find the fewest number of final exam slots that are needed to avoid any conflicts

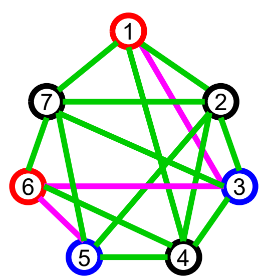
Coloring: Application 3



Answer is 4



K_4



Bipartite

7 is not connected to 4