Discrete Mathematic

Chapter 4: Advanced Counting Techniques

4.1 Recurrence Relations

4.2 Solving Linear Recurrence Relations

4.4 Generating Functions

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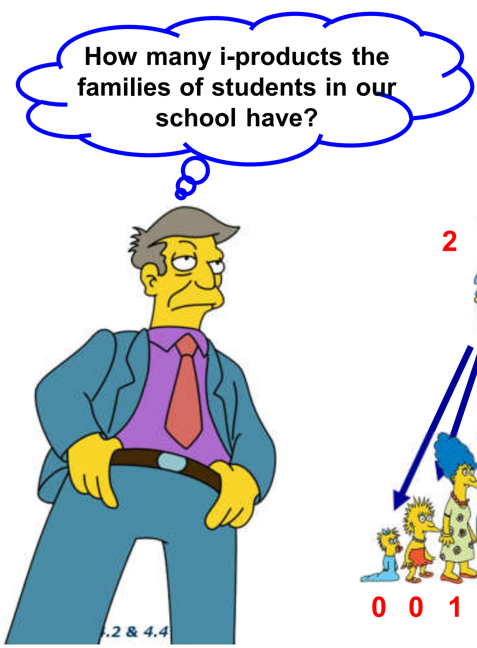
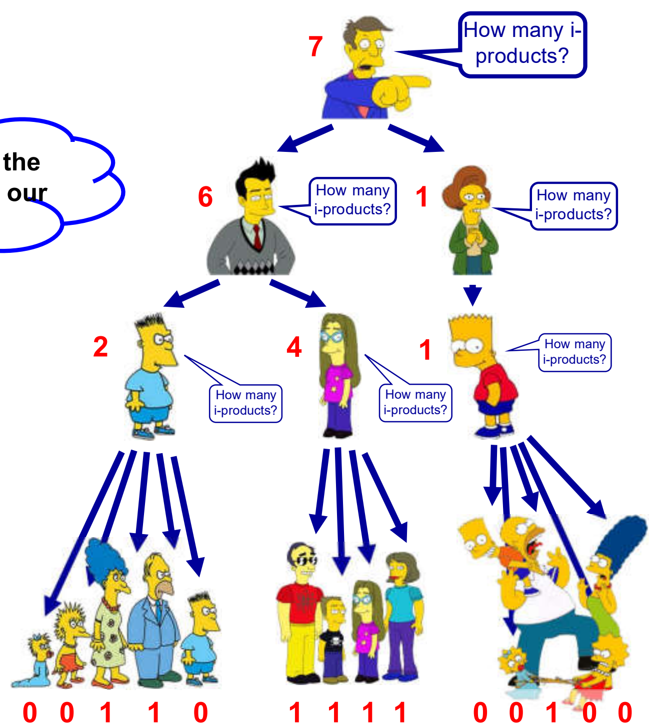
Agenda

- Recurrence Relations
- Modeling with Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations with Constant Coefficients
- Generating Functions
- Useful Facts About Power Series
- Extended Binomial Coefficient
- Extended Binomial Theorem
- Counting Problems and Generating Functions
- Using Generating Functions to Solve Recurrence Relations

Recursion





Recurrence Relations

- A **recurrence relation** for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements ($a_0, \ldots, a_{n-1}$)
- A sequence is called a **solution** of a recurrence relation **if** its terms satisfy the recurrence relation

Recurrence Relations
Example 1

- Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2, 3, 4, . . . , and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
- From the recurrence relation:
  - $a_2 = a_1 - a_0 = 5 - 3 = 2$
  - $a_3 = a_2 - a_1 = 2 - 5 = -3$

Recurrence Relations
Example 2

- Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}, \text{ where } n \geq 2$$

- Which of the following are solutions?
  - $a_n = 3n$ ✓
    - $2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$
  - $a_n = 2^n$ ✗
    - $2a_{n-1} - a_{n-2} = 2(2^{n-1}) - n^{n-2} = 2^n \neq a_n$
  - $a_n = 5$ ✓
    - $2a_{n-1} - a_{n-2} = 2 \times 10 - 5 = 5 = a_n$

Recurrence Relations

- The **initial conditions** for a sequence specify the terms that precede the first term where the recurrence relation takes effect
  - For example
    $a_n = a_{n-1} + a_{n-2}$, what is the value of a_3?
    Answer depends on a_0 and a_1 (initial conditions)
    - $a_0 = 3$ and $a_1 = 5$: $a_2 = 8$, $\mathbf{a_3 = 13}$
    - $a_0 = 1$ and $a_1 = 2$: $a_2 = 3$, $\mathbf{a_3 = 5}$
- A **sequence** is determined uniquely by
  - Recurrence relation
  - Initial conditions

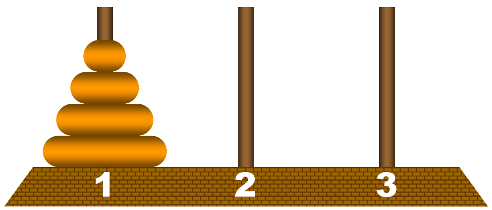
Modeling with Recurrence Relations
Compound Interest

- Growth of saving in a bank account with r% interest per given period
  - $\mathbf{S_n = S_{n-1} + r \cdot S_{n-1} = (r+1) \cdot S_{n-1}}$
- Example:
  - Suppose that a person deposits $10,000 in a savings account at a bank yielding 11 % per year with interest compounded annually. How much will be in the account after 30 years?
  - $S_{30} = 1.11S_{29} = 1.11(1.11S_{28}) = \ldots = (1.11)^{30}\ 10{,}000$

Modeling with Recurrence Relations

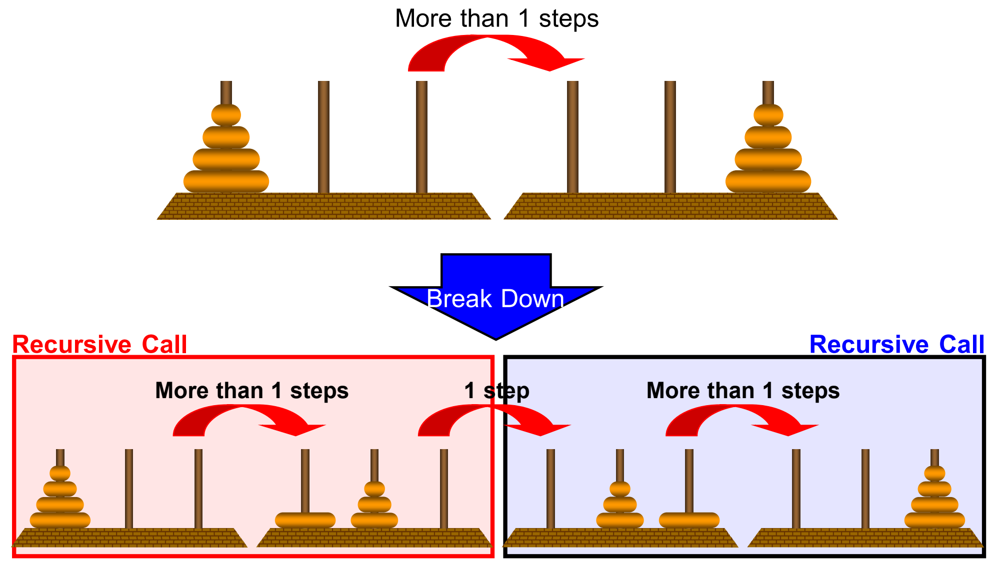
Tower of Hanoi

- **Objective**
  - Get all disks from peg 1 to peg 3
- **Rules**
  - Only move 1 disk at a time
  - Never put a larger disk on a smaller one



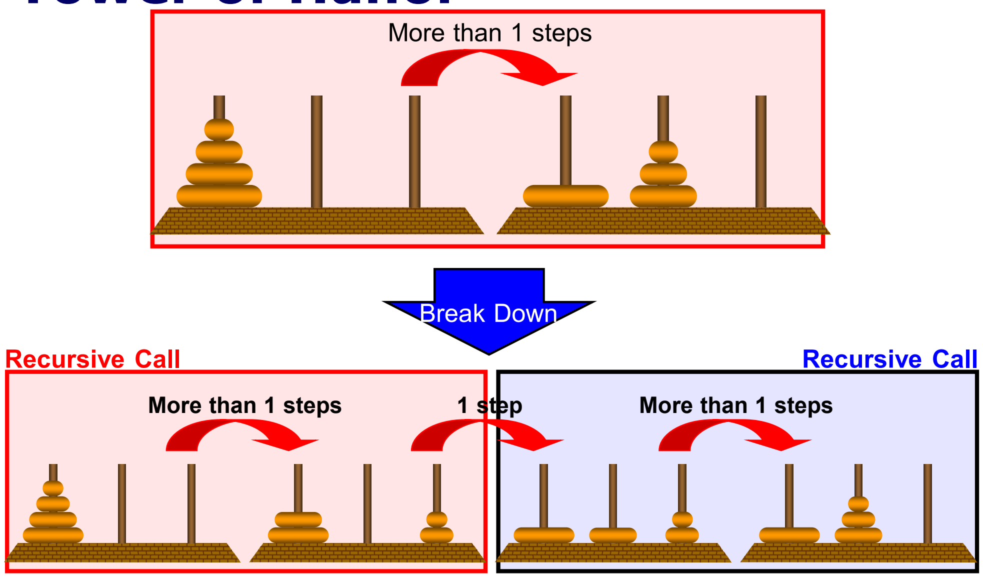
Modeling with Recurrence Relations

Tower of Hanoi



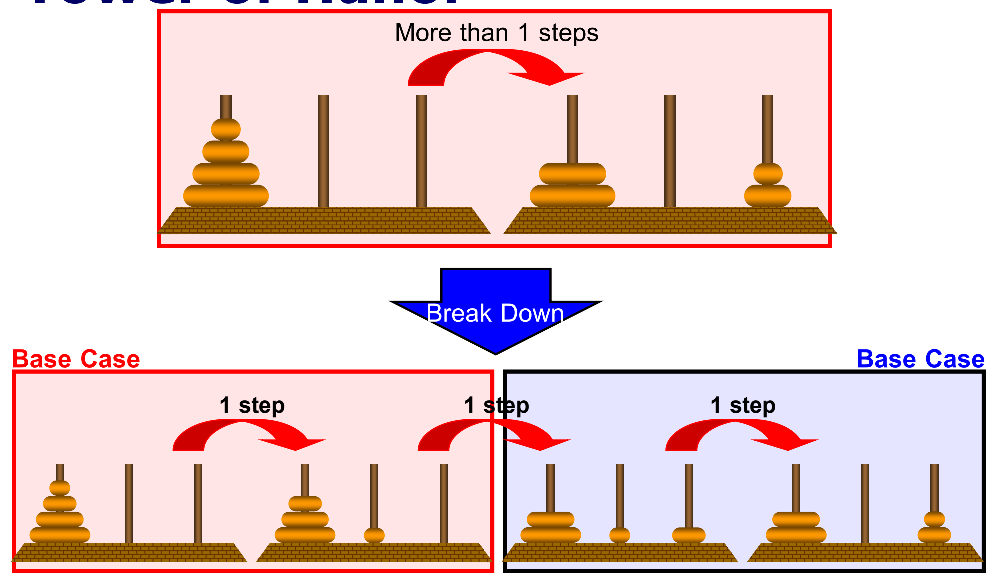
Modeling with Recurrence Relations

Tower of Hanoi



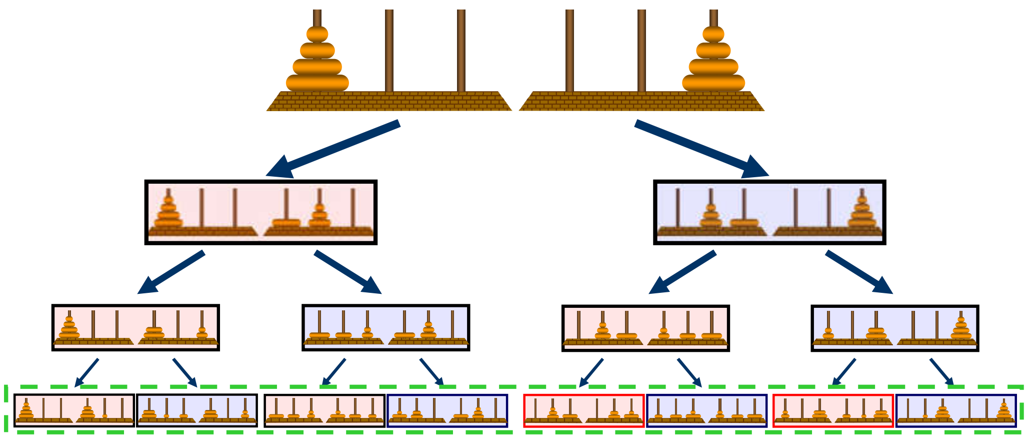
Modeling with Recurrence Relations

Tower of Hanoi



Modeling with Recurrence Relations

Tower of Hanoi



The solution

Modeling with Recurrence Relations

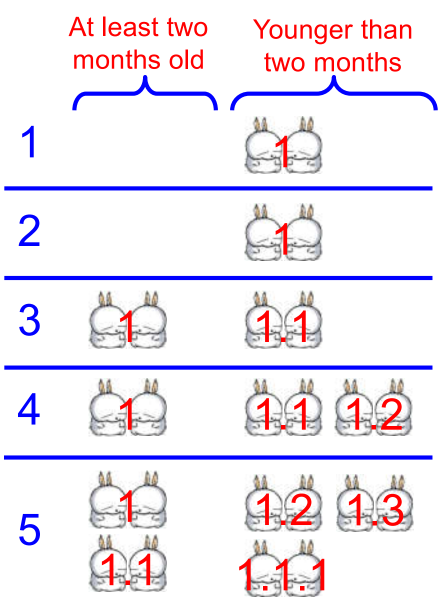
Tower of Hanoi

- Let H_n be the number of moves for a stack of n disks.
- Strategy:
  - Move top n−1 disks (H_{n-1} moves)
  - Move bottom disk (1 move)
  - Move top n−1 to bottom disk (H_{n-1} moves)
- $H_n = 2H_{n-1} + 1$

Modeling with Recurrence Relations

Fibonacci (Rabbits) Numbers

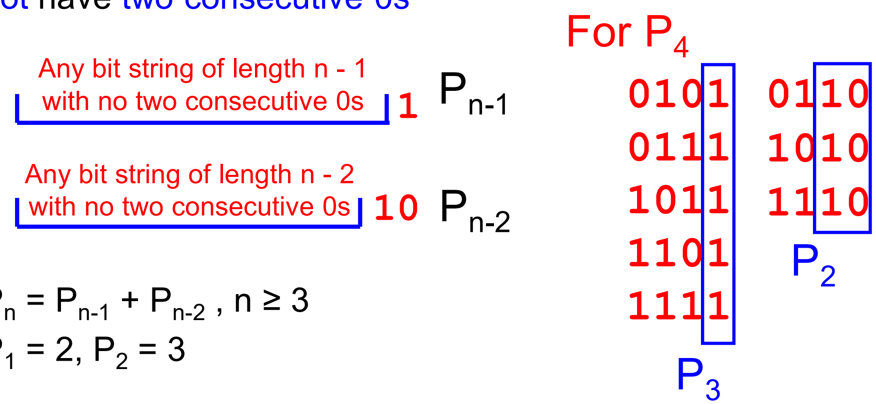
- A young pair of rabbits (one of each sex) is placed on an island
- A pair of rabbits does not breed until they are 2 months old
- After they are 2 months old, each pair of rabbits produces another pair each month
- $P_n = P_{n-1} + P_{n-2}$



Modeling with Recurrence Relations

Example 1

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s.
- Let P_n denote the number of bit strings of length n that do not have two consecutive 0s



- $P_n = P_{n-1} + P_{n-2}$, n ≥ 3
- $P_1 = 2$, $P_2 = 3$

Example 2

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have three consecutive 0s.
- Let P_n denote the number of bit strings of length n that do not have three consecutive 0s

Any bit string of length n - 1 with no three consecutive 0s | 1 → P_{n-1}

Any bit string of length n - 2 with no three consecutive 0s | 10 → P_{n-2}

Any bit string of length n - 3 with no three consecutive 0s | 100 → P_{n-2}

For P_4

P_3	P_2	P_1
001 1	00 10	0 100
010 1	01 10	1 100
011 1	10 10	
100 1	11 10	
101 1		
110 1		
111 1		

- $P_n = P_{n-1} + P_{n-2} + P_{n-3}$, $n \geq 4$
- $P_1 = 2$, $P_2 = 4$, $P_3 = 7$

Solving Linear Recurrence Relations

- Given $P_n = P_{n-1} + P_{n-2}$, what is P_{100}?
- It is not easy to calculate
- Need a **better solution** which is **not in relation form**
  E.g. $P_n = n*10 - 1$

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence of Degree k with Constant Coefficients is a recurrence of the form

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} = \sum_{i=1}^{k} c_i a_{n-i}$$

where the c_i are all real numbers, and $c_k \neq 0$

  - **Li**near: the power of all a_i term is one
  - **Ho**mogeneous: no constant term (no team without a_i)
  - **Re**currence: a sequence $\{a_n\}$ which a_n in terms of $a_{n-1}, a_{n-2}, \ldots$
  - Degree **k**: refer to k previous terms a_{n-k}
  - **Co**nstant **Co**efficients: $c_1, c_2, \ldots$ independent from n
- The short name is "**k-LiHoReCoCo**"

Example

- $M_n = M_{n-1} + (1.11)M_{n-1}$
  - 1-LiHoReCoCo
- $P_n = P_{n-1} + P_{n-2}$
  - 2-LiHoReCoCo
- $a_n = a_{n-5}$
  - 5-LiHoReCoCo
- $a_n = a_{n-1} + (a_{n-2})^2$
  - Not linear
- $H_n = 2H_{n-1} + 1$
  - Not homogeneous
- $B_n = nB_{n-1}$
  - Non-constant coefficient (n is a variable)

k-LiHoReCoCo

- **Li**near: the power of all a_i term is one
- **Ho**mogeneous: no constant term (no team without a_i)
- **Re**currence: a sequence $\{a_n\}$ which a_n in terms of $a_{n-1}, a_{n-2}, \ldots$
- Degree **k**: refer to k previous terms a_{n-k}
- **Co**nstant **Co**efficients: $c_1, c_2, \ldots$ independent from n

Solving 2-LiHoReCoCos

- Given 2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and $a_0 = c$ and $a_1 = d$
- Assume s_n and t_n be the solution, $a_n = s_n$ and $a_n = t_n$
  - $s_n = c_1 s_{n-1} + c_2 s_{n-2}$ and $t_n = c_1 t_{n-1} + c_2 t_{n-2}$
- For constants w_1 and w_2

$$\underbrace{w_1 s_n + w_2 t_n}_{a_n} = w_1(c_1 s_{n-1} + c_2 s_{n-2}) + w_2(c_1 t_{n-1} + c_2 t_{n-2})$$
$$= c_1\underbrace{(w_1 s_{n-1} + w_2 t_{n-1})}_{a_{n-1}} + c_2\underbrace{(w_1 t_{n-2} + w_2 t_{n-2})}_{a_{n-2}}$$

- Therefore, $a_n = w_1 s_n + w_2 t_n$ is a **solution**

Solving 2-LiHoReCoCos

- By considering 1-LiHoReCoCo, $a_n = c\, a_{n-1}$
- Obviously, the general solution is $a_n = c^n a_0$
- Therefore, the solution of the form may be $a_n = r^n$
- Substitute $a_n = r^n$ to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $r^2 - c_1 r - c_2 = 0$ or $r = 0$
  - $r = 0$ is a special case since $a_n = 0$
- $r^2 - c_1 r - c_2$ is called **characteristic equation**

Solving 2-LiHoReCoCos

- Given 2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and $a_0 = c$ and $a_1 = d$
- Assume $a_n = w_1 r_1^n + w_2 r_2^n$ for r_1 and r_2 are different and some constants w_1, w_2
- We know that $r_1^2 - c_1 r_1 - c_2 = 0$ and $r_2^2 - c_1 r_2 - c_2 = 0$
- Characteristic Equation: $r^2 - c_1 r - c_2 = 0$
- Characteristic Roots: r_1 and r_2
- w_1 and w_2 can be calculated by using c and d

$$\begin{cases} a_0 = c = w_1 r_1^0 + w_2 r_2^0 \\ a_1 = d = w_1 r_1^1 + w_2 r_2^1 \end{cases}$$

Solving 2-LiHoReCoCos

Theorem

- Consider an arbitrary 2-LiHoReCoCo:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- By substituting $a_n = r^n$, we have the characteristic equation:

$$r^2 - c_1 r - c_2 = 0$$

- If there has two different roots r_1 and r_2, then

$$a_n = w_1 r_1^n + w_2 r_2^n$$

for $n \geq 0$ and some constants w_1, w_2

Solving 2-LiHoReCoCos

- **Proof**

  Given r_1, r_2 are the characteristic root

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \Leftrightarrow \quad a_n = w_1 r_1^n + w_2 r_2^n$$

  where and w_1, w_2 are constants

  - **Two steps** for the proof
    1. Show if $a_n = w_1 r_1^n + w_2 r_2^n$, $\{a_n\}$ is a solution of the recurrence relation
    2. Show if $\{a_n\}$ is the solution of the recurrence relation, $a_n = w_1 r_1^n + w_2 r_2^n$ for some w_1 and w_2

Solving 2-LiHoReCoCos

- **Step 1**
  Show if $a_n = w_1 r_1^n + w_2 r_2^n$, $\{a_n\}$ is a solution of the recurrence relation

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1\left(w_1 r_1^{n-1} + w_2 r_2^{n-1}\right) + c_2\left(w_2 r_1^{n-2} + w_2 r_2^{n-2}\right) \\ &= w_1 r_1^{n-2}\left(c_1 r_1 + c_2\right) + w_2 r_2^{n-2}\left(c_1 r_2 + c_2\right) \\ &= w_1 r_1^{n-2} r_1^2 + w_2 r_2^{n-2} r_2^2 \\ &= w_1 r_1^n + w_2 r_2^n \\ &= a_n \end{aligned}$$

(r_1 and r_2 are the solution of $r^2 - c_1 r - c_2 = 0$)

Solving 2-LiHoReCoCos

- **Step 2**
  Show if $\{a_n\}$ is the solution of the recurrence relation, $a_n = w_1 r_1^n + w_2 r_2^n$ for some w_1 and w_2
  - Suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold
  - We want to show that there are constants w_1 and w_2 such that the sequence $\{a_n\}$ with $a_n = w_1 r_1^n + w_2 r_2^n$ satisfies these same initial conditions
    - $a_0 = C_0 = w_1 + w_2$ and $a_1 = C_1 = w_1 r_1 + w_2 r_2$
  - By solving these two equations:

    $$w_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2} \qquad w_2 = \frac{C_0 r_2 - C_1}{r_1 - r_2}$$

  - When $r_1 \neq r_2$, $\{a_n\}$ with $w_1 r_1^n + w_2 r_2^n$ satisfy the 2 initial conditions

Solving 2-LiHoReCoCos

- We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$
- Because there is a unique solution of 2-LiHoReCoCo with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all nonnegative integers n
- We have completed the proof

Solving 2-LiHoReCoCos

Example 1

2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$,
Characteristic Equation: $r^2 - c_1 r - c_2 = 0$
$a_n = w_1 r_1^n + w_2 r_2^n$ (r_1 and r_2 are different)

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$
- Characteristic Equation: $r^2 - r - 2 = 0$
- Characteristic Root:
  - $r = (1 \pm 3) / 2$
  - $r = 2$ or $r = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Therefore, $a_n = w_1 2^n + w_2 (-1)^n$
- By using $a_0 = 2$, $a_1 = 7$
  - $a_0 = 2 = w_1 2^0 + w_2 (-1)^0$
  - $a_1 = 7 = w_1 2^1 + w_2 (-1)^1$
  - $w_1 = 3$ and $w_2 = 1$
- Therefore, $a_n = 3 \cdot 2^n - (-1)^n$

Solving 2-LiHoReCoCos

Example 2

2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$,
Characteristic Equation: $r^2 - c_1 r - c_2 = 0$
$a_n = w_1 r_1^n + w_2 r_2^n$ (r_1 and r_2 are different)

- Find an explicit formula for the Fibonacci numbers
- Recall $f_n = f_{n-1} + f_{n-2}$
  - Characteristic equation: $r^2 - r - 1 = 0$
  - Characteristic roots: $r_1 = (1 + \sqrt{5}) / 2$ $\quad r_2 = (1 - \sqrt{5}) / 2$

$$f_n = w_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + w_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

- Initial conditions $f_0 = 0$ and $f_1 = 1$

$$f_0 = 0 = w_1 + w_2$$

$$f_1 = 1 = w_1 \left(\frac{1 + \sqrt{5}}{2} \right) + w_2 \left(\frac{1 - \sqrt{5}}{2} \right) \qquad w_1 = \frac{1}{\sqrt{5}} \qquad w_2 = -\frac{1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Solving 2-LiHoReCoCos with two same roots

Theorem

- Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0
- A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = w_1 r_0^n + w_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where w_1 and w_2 are constants

Solving 2-LiHoReCoCos with

Example 1

2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$,
Characteristic Equation: $r^2 - c_1 r - c_2 = 0$
$a_n = w_1 r_0^n + w_2 n r_0^n$

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?
- Characteristic equation: $r^2 - 6r + 9 = 0$
- Only one characteristic root: $r = 3$
- Hence, the solution to this recurrence relation is

$$a_n = w_1 3^n + w_2 n 3^n$$

for some constants α_1 and α_2

- By using the initial conditions,

$a_0 = 1 = w_1$, $a_1 = 6 = 3w_1 + 3w_2$, so $w_1 = 1$ and $w_2 = 1$

- Consequently, $a^n = 3^n + n3^n$

Solving 2-LiHoReCoCos

Summary

- Given : $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $a_0 = c$ and $a_1 = d$
  1. **Characteristic equation:** $r^2 - c_1 r - c_2 = 0$

  2a. If Characteristic Root (r_1 and r_2) are different
  1. $a_n = w_1 r_1^n + w_2 r_2^n$ is the solution
  2. Use $a_0 = w_1 + w_2 = c$ and $a_1 = w_1 r_1 + w_2 r_2 = d$ to solve w_1 and w_2

  2b. If Characteristic Root (r_1 and r_2) are the same
  1. $a_n = w_1 r^n + w_2 n r^n$ is the solution
  2. Use $a_0 = w_1 = c$ and $a_1 = w_1 r_1 + w_2 r_2 = d$ to solve w_1 and w_2

☺ Small Exercise ☺

- What is the solution of the recurrence relation $a_n = - a_{n-1} + 6a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 5$?
- What is the solution of the recurrence relation $a_n = - 2a_{n-1} - a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = -6$?

☺ Small Exercise ☺

- the recurrence relation: $a_n = - a_n + 6a_{n-2}$
- Initial conditions $a_0 = 0$ and $a_1 = 5$
- Characteristic Equation: $r^2 + r - 6 = 0$
  $(r + 3)(r - 2) = 0$
- Characteristic Root: $r_1 = -3,\ r_2 = 2$
- Therefore, $a^n = w_1 (-3)^n + w_2 (2)^n$
- Using the initial condition
  - $a_0 = 0 = w_1 + w_2$
  - $a_1 = 5 = -3w_1 + 2w_2$
  - $w_1 = -1,\ w_2 = 1$
- Therefore, $a^n = - (-3)^n + (2)^n$

☺ Small Exercise ☺

- the recurrence relation: $a_n = - 2a_n - a_{n-2}$
- Initial conditions $a_0 = 5$ and $a_1 = -6$
- Characteristic Equation: $r^2 + 2r + 1 = 0$
  $(r + 1)(r + 1) = 0$
- Characteristic Root: $r_1 = -1$
- Therefore, $a^n = w_1 (-1)^n + w_2\, n\, (-1)^n$
- Using the initial condition
  - $a_0 = 5 = w_1$
  - $a_1 = -6 = -w_1 - w_2$
  - $w_1 = 5,\ w_2 = 1$
- Therefore, $a^n = 5 (-1)^n + n (-1)^n$

Solving k-LiHoReCoCos

- k-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$

2-kiHoReCoCo

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$

- **Characteristic Equation** is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$

$r^2 - c_1 r - c_2 = 0$

- **Theorem**
  If there are k distinct roots r_i , then the solutions to the recurrence are of the form:

$$a_n = \sum_{i=1}^{k} w_i r_i^n$$

$a_n = w_1 r_1^n + w_2 r_2^n$

  for all $n \geq 0$, where the w_i are constants

Solving k-LiHoReCoCos
Example

- Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

  with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.
- The characteristic equation is:

$$r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$$

  - The characteristic roots are $r = 1$, $r = 2$, and $r = 3$
- $a_n = w_1 1^n + w_2 2^n + w_3 3^n$
- By using the initial conditions
  - $a_0 = 2 = w_1 + w_2 + w_3$
  - $a_1 = 5 = w_1 + w_2 \times 2 + w_3 \times 3$
  - $a_2 = 15 = w_1 + w_2 \times 4 + w_3 \times 9$
  - Therefore, $w_1 = 1$, $w_2 = -1$ and $w_3 = 2$
- As a result, $a_n = 1 - 2^n + 2 \times 3^n$

Solving k-LiHoReCoCos with same roots

- Let $c_1, c_2, ..., c_k$ be real numbers
- Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

  has t distinct roots $r_1, r_2, ..., r_t$ with multiplicities $m_1, m_2, ..., m_t$
  - i.e. r_i appear m_i times
  - $m_1 + m_2 + \cdots + m_t = k$

Solving k-LiHoReCoCos with same roots

Special case for k=2, One distinct root

$a_n = w_1 r_0^n + w_2 n r_0^n$

$a_n = (w_1 + w_2 n) r_0^n$

- A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$$

- If and only if

Multiplicities for r_i

$$\begin{aligned} a_n &= (w_{1,0} + w_{1,1} n + ... + w_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (w_{2,0} + w_{2,1} n + ... + w_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ ... + (w_{t,0} + w_{t,1} n + ... + w_{t,m_t-1} n^{m_t-1}) r_t^n \\ &= \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i-1} w_{i,j} n^j \right) r_i^n \end{aligned}$$

No. of distinct roots

  for $n = 0,1,2, ...$, where $w_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$

Solving k-LiHoReCoCos with s

Example

$$a_n = (w_{1,0} + w_{1,1}n + \ldots + w_{1,m_1-1}n^{m_1-1})r_1^n + (w_{2,0} + w_{2,1}n + \ldots + w_{2,m_2-1}n^{m_2-1})r_2^n + \ldots + (w_{t,0} + w_{t,1}n + \ldots + w_{t,m_t-1}n^{m_t-1})r_t^n$$

- Find the solution to the recurrence relation

$$H_n = -H_{n-1} + 3H_{n-2} + 5H_{n-3} + 2H_{n-4}$$

with the initial conditions $H_0 = 1, H_1 = 0, H_2 = 1, H_3 = 2$

- **Characteristic equation**: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

$$(x-2)(x+1)^3 = 0$$

- **Roots**: $-1, -1, -1, 2$
- Therefore: $H_n = (c_1 + c_2 n + c_3 n^2)(-1)^n + c_4 2^n$
- By initial conditions:

$$\begin{cases} H_0 = c_1 + c_4 = 1 \\ H_1 = -c_1 - c_2 - c_3 + 2c_4 = 0 \\ H_2 = c_1 + 2c_2 + 4c_3 + 4c_4 = 1 \\ H_3 = -c_1 - 3c_2 - 9c_3 + 8c_4 = 2 \end{cases}$$

$$c_1 = \frac{7}{9}, c_2 = -\frac{1}{3}, c_3 = 0, c_4 = \frac{2}{9}$$

$$H_n = \frac{7}{9}(-1)^n - \frac{1}{3}n(-1)^n + \frac{2}{9}2^n$$

Solving k-LiHoReCoCos

Summary

- Given : $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ and $a_i = c_i$, where $i = 1, 2, \ldots, k$

1. Characteristic equation: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$

2. Characteristic Root ($r_1, r_2, \ldots, r_k$)

   1. $a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1} w_{i,j} n^j\right) r_i^n$ is the solution of k-LiHoReCoCos where m_i is the multiplicity of r_i

   2. solve w_i by $a_p = c_p = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1} w_{i,j} p^j\right) r_i^p$ where p = 1, 2, ..., k

☺ Small Exercise ☺

- What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 0$, $a_1 = 8$ and $a_2 = 4$?

$$a_n = (w_{1,0} + w_{1,1}n + \ldots + w_{1,m_1-1}n^{m_1-1})r_1^n + (w_{2,0} + w_{2,1}n + \ldots + w_{2,m_2-1}n^{m_2-1})r_2^n + \ldots + (w_{t,0} + w_{t,1}n + \ldots + w_{t,m_t-1}n^{m_t-1})r_t^n$$

☺ Small Exercise ☺

- the recurrence relation: $a_n = a_{n-1} + a_{n-2} - a_{n-3}$
- Initial conditions $a_0 = 0$, $a_1 = 8$ and $a_2 = 4$
- Characteristic Equation: $r^3 - r^2 - r + 1 = 0$

$$(r-1)(r-1)(r+1) = 0$$

- Characteristic Root: $r_1 = 1, r_2 = 1, r_3 = -1$
- Therefore, $a^n = (c_1 + c_2 n)(1)^n + c_3(-1)^n$
- Using the initial condition
  - $a_0 = 0 = c_1 + c_3$
  - $a_1 = 8 = c_1 + c_2 - c_3$
  - $a_2 = 4 = c_1 + 2c_2 + c_3$
  - $c_1 = 3, c_2 = -3, c_3 = 2$
- Therefore, $a^n = 3(1-n)(1)^n + 2(-1)^n$

Solving LiNoReCoCos

- Linear nonhomogeneous recurrence of degree k with constant coefficients (k-LiNoReCoCos) contain some terms $F(n)$ that depend only on n but not a_i
- **General form**:

$$\underbrace{a_n = c_1 a^{n-1} + \dots + c_k a^{n-k}} + F(n)$$

Associated Homogeneous Recurrence Relation

Solving LiNoReCoCos

- If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

- Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

Solving LiNoReCoCos

- **Proof**
  - As $\{a_n^{(p)}\}$ is a particular solution for LiNoReCoCos
  - Suppose that $\{b_n\}$ is an another solution

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \dots + c_k a_{n-k}^{(p)} + F(n)$$

$$- \quad b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + F(n)$$

$$b_n - a_n^{(p)} = c_1\left(b_{n-1} - a_{n-1}^{(p)}\right) + c_2\left(b_{n-2} - a_{n-2}^{(p)}\right) + \dots + c_k\left(b_{n-k} - a_{n-k}^{(p)}\right)$$

$$a_n^{(h)} = c_1 a_{n-1}^{(h)} + c_2 a_{n-2}^{(h)} + \dots + c_k a_{n-k}^{(h)}$$

  - $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, named $\{a_n^{(h)}\}$
  - Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Solving LiNoReCoCos

Example 1

$a_n = 3a_{n-1} + 2n$
$a_1 = 3$

- Find all solutions to $a_n = 3a_{n-1} + 2n$, which solution has $a_1 = 3$?
- Notice this is a 1-LiNoReCoCo.
- Its associated 1-LiHoReCoCo
  - $a_n = 3a_{n-1}$ and root is 3
  - Solution is $a_n^{(h)} = c3^n$
- The solutions of LiNoReCoCo are in the form

$$a_n = a_n^{(p)} + a_n^{(h)}$$

Next step

Solving LiNoReCoCos
Example

$a_n = 3a_{n-1}+2n$
$a_1 = 3$

- If $F(n)$ is a degree-u polynomial in n, a degree-u polynomial should be tried as the particular solution $a_n^{(p)}$
- Now, $F(n) = 2n$
  - Try $a_n^{(p)} = cn + d$, c and d are constants

$$a_n = 3a_{n-1} + 2n$$
$$cn+d = 3(c(n-1)+d) + 2n$$
$$(2c+2)n + (3c-2d) = 0$$
$$c = -1 \text{ and } d = -3/2$$

  - Solution is: $a_n^{(p)} = -n - 3/2$

Solving LiNoReCoCos
Example

$a_n = 3a_{n-1}+2n$
$a_1 = 3$

- Therefore, we have

$$a_n = a_n^{(p)} + a_n^{(h)}$$
$$= -n - \frac{3}{2} + c3^n$$

$$a_n^{(p)} = -n - \frac{3}{2}$$
$$a_n^{(h)} = c3^n$$

- By using $a_1 = 3$
  - $3 = -1 - 3/2 + 3c$
  - $c = 11/6$
- As a result, $a_n = -n - \frac{3}{2} + \frac{11 \cdot 3^n}{6}$

Solving LiNoReCoCos
Particular Solution

- Suppose $\{a_n\}$ satisfies the LiNoReCoCo $a_n = \left(\sum_{i=1}^{k} c_i a_{n-i}\right) + F(n)$ where c_i $(i = 1,2,\ldots k)$ are real numbers and

$$F(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n + b_0)s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers

- When s **is not a root** of the characteristic equation of the associated linear homogeneous RR, there is a particular solution ($a_n^{(p)}$) of the form

$$(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n$$

- When s **is a root** of the characteristic equation of the associated linear homogeneous RR with multiplicity m, there is a particular solution ($a_n^{(p)}$) of the form

$$n^m(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n$$

Solving LiNoReCoCos
Example

$F(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n + b_0)s^n$

s is not a root $(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n$

s is a root $n^m(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n$

- What form do[es the particular solution of the] nonhomogen[eous RR] $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when ...
- Consider the associated homogeneous RR:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

- Characteristic Equation

$$r^2 - 6r + 9 = (r-3)^2 = 0$$

- Characteristic Root is 3, of multiplicity m=2

- $F(n) = 3^n$
  $n^2(p_0)3^n$
- $F(n) = n3^n$
  $n^2(p_1 n + p_0)3^n$
- $F(n) = n^2 2^n$
  $(p_2 n^2 + p_1 n + p_0)2^n$
- $F(n) = (n^2 + 1)3^n$
  $n^2(p_2 n^2 + p_1 n + p_0)3^n$

Solving LiNoReCoCos : Pa[rticular Solution]

Example 2

$F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0) s^n$

s is not a root $(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n$

s is a root $n^m (p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n$

- Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k$$

- a_n satisfies the linear nonhomogeneous RR

$$a_n = a_{n-1} + n$$

- Associated linear homogeneous RR is $a_n = a_{n-1}$
- Root is 1. The solution is $a_n^{(h)} = c(1)^n$, c is a constant
- Since $F(n) = n = n \times (1)^n$, and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous RR
- So the particular solution has the form $n(p_1 n + p_0)$

Solving LiNoReCoCos : Particular Solu[tion]

Example 2

$a_n = a_{n-1} + n$

$a_n^{(p)} = n(p_1 n + p_0)$

- By solving

$$p_1 n^2 + p_0 n = p_1 (n - 1)^2 + p_0 (n - 1) + n$$

- We have $p_0 = p_1 = 1/2$
- Recall, $a_n = a_n^{(p)} + a_n^{(h)}$

$$a_n = n(n+1)/2 + c$$

$a_n^{(h)} = c$

$a_n^{(p)} = n(n+1)/2$

- By using $a_1 = 1$, so $c = 0$
- Therefore,

$$a_n = \frac{n(n+1)}{2}$$

Solving Linear Recurrence Relations

Summary

- **k-LiHoReCoCos with m same roots (*without F(x)*)**
  - Find the root of characteristic equation
  - $a_n = (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$
  - Use initial terms to find alphas
- **k-LiNoReCoCos with m same roots (*with F(x)*)**
  - Find the solution of characteristic equation of Associated linear homogeneous RR $a_n^{(h)}$
  - Find the particular solution of LiNoReCoCo using

$$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n$$

  - Finally $a_n^{(p)} + a_n^{(h)}$
  - Use initial terms to find alphas

☺ Small Exercise ☺

- Find all solutions to $a_n = 7a_{n-1} + (2n^2+2)3^n$, which solution has $a_1 = 10$?

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Its associated 1-LiHoReCoCo
  - $a_n = 7a_{n-1}$ and root is 7
  - Solution is $a_n^{(h)} = c7^n$
- The solutions of 1-LiNoReCoCo are in the form

$$a_n = a_n^{(p)} + a_n^{(h)}$$

- Need to do is **find one** $a_n^{(p)}$

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Now, $F(n) = (2n^2+2)3^n$
  - $a_n^{(p)} = (an^2 + bn + c)3^n$

$$a_n = 7a_{n-1} + (2n^2+2)3^n$$
$$(an^2+bn+c)3^n = 7(a(n-1)^2+b(n-1)+c)3^{n-1} + (2n^2+2)3^n$$
$$3an^2+3bn+3c = 7an^2 - 14an + 7a + 7bn - 7b + 7c + 6n^2 + 6$$
$$0 = 4an^2 - 14an + 7a + 4bn - 7b + 4c + 6n^2 + 6$$
$$0 = n^2(4a+6)+n(4b-14a) +(4c +7a-7b+6)$$

$4a+6=0$ $4b-14a=0$ $4c +7a-7b+6=0$

$a=-3/2$ $b=-21/4$ $c=-129/16$

$$a_n^{(p)} = (-3n^2/2 - 21n/4 - 129/6)3^n$$

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Therefore, we have

$$a_n = a_n^{(p)} + a_n^{(h)}$$
$$= (-3n^2/2 - 21n/4 - 129/6)3^n + c7^n$$

$a_n^{(p)} = (-3n^2/2 - 21n/4 - 129/6)3^n$

$a_n^{(h)} = c7^n$

- By using $a_0 = 10$
  - $a_0 = 10 = -129/6 + c$
  - $c = 189/6$
- As a result,

$$a_n = (-3n^2/2 - 21n/4 - 129/6)3^n + 189 \cdot 7^n/6$$

Generating Functions

- **Generating functions** (**G(x)**) are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series
- **Generating function** for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0x^0 + a_1x^1 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

Example

- What is the generating function for the following sequence?
  - **{0, 2, …, 2k,…}**

$$0 + 2x + \ldots + 2k \cdot x^k + \ldots = \sum_{k=0}^{\infty} 2^k x^k$$

  - **{1, 1, 1, 1, 1}**

$$1 + x + x^2 + x^3 + x^4 = \sum_{k=0}^{4} x^k$$

Useful Facts About Power Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

- $f(x) = \frac{1}{1-x}$

  is generating function of the sequence 1,1,1,1, ...

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

- $f(x) = \frac{1}{1-ax}$

  is generating function of the sequence 1, a, a^2, a^3, ...

Useful Facts About Power Series

- Given: $f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k$

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k$$

$$= (a_0x^0 + a_1x^1 + \ldots)(b_0x^0 + b_1x^1 + \ldots)$$

$$= x^0(a_0b_0) + x^1(a_0b_1 + a_1b_0) + x^2(a_0b_2 + a_1b_1 + a_2b_0) + \ldots$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (a_j b_{k-j}) x^k$$

Useful Facts About Power Series

Example

- Let $h(x) = \frac{1}{(1-x)^2}$,
- Find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $h(x) = \sum_{k=0}^{\infty} a_k x^k$

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (a_j b_{k-j}) x^k$$

$$h(x) = \frac{1}{(1-x)^2} = \frac{1}{(1-x)} \frac{1}{(1-x)}$$

$$= \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right)$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} x^k$$

$$= \sum_{k=0}^{\infty} (k+1) x^k$$

$a_k = k+1$

Counting Problems and Generating Functions

- How can we solve the counting problems, including the recurrence relation, by using the Generating Functions?

$$G(x) = a_0x^0 + a_1x^1 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

Example 1

- Find the number of solutions of $e_1 + e_2 + e_3 = n$ when $n = 17$, where e_1, e_2, e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, $4 \le e_3 \le 7$
- By considering this generating function for the sequence $\{a_n\}$, where a_n is the number of solution for n

$$\sum_{k=0}^{\infty} a_kx^k = \begin{pmatrix} (x^2 + x^3 + x^4 + x^5)\cdot \\ (x^3 + x^4 + x^5 + x^6)\cdot \\ (x^4 + x^5 + x^6 + x^7) \end{pmatrix}$$

- As $n = 17$, a_{17}, which is the coefficient of x^{17}, is the solution
- Answer is 3

Example 2

- In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?
- By considering this generating function for the sequence $\{a_n\}$, where a_n is the number of solution for n

$$\sum_{k=0}^{\infty} a_kx^k = (x^2 + x^3 + x^4)(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)$$

- The coefficient of x^8 is 6

Example 3

- Solve the recurrence relation $a_k = 3a_{k-1}$ for $k=1, 2, 3, \ldots$ and initial condition $a_0=2$
- Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_kx^k$$

$$G(x) = \sum_{k=0}^{\infty} 3a_{k-1}x^k$$

$$G(x) = a_0 + \sum_{k=1}^{\infty} 3a_{k-1}x^k$$

$$G(x) = a_0 + 3x\sum_{k=1}^{\infty} a_{k-1}x^{k-1}$$

$$G(x) = a_0 + 3xG(x)$$

$$G(x) = \frac{2}{1-3x}$$

$$G(x) = \sum_{k=0}^{\infty} 2\cdot 3^k x^k$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}}$$

$$a_k = 2\cdot 3^k$$

Counting Problems and Generating Functions
Example 4

- Solve the recurrence relation $a_k = -a_{k-1} + 6a_{k-2}$ with initial conditions $a_0 = 0$ and $a_1 = 5$
- Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0 x^0 + a_1 x^1 + \sum_{k=2}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1 x + \sum_{k=2}^{\infty} (-ak_{-1} + 6a_{k-2}) x^k$$

$$G(x) = a_0 + a_1 x - \sum_{k=2}^{\infty} a_{k-1} x^k + 6 \sum_{k=2}^{\infty} a_{k-2} x^k$$

Counting Problems and Generating Functions
Example 4

$$G(x) = a_0 + a_1 x - \sum_{k=2}^{\infty} a_{k-1} x^k + 6 \sum_{k=2}^{\infty} a_{k-2} x^k$$

$$G(x) = a_0 + a_1 x + a_0 x - a_0 x - x \sum_{k=2}^{\infty} a_{k-1} x^{k-1} + 6x^2 \sum_{k=2}^{\infty} a_{k-2} x^{k-2}$$

$$G(x) = a_0 + a_1 x + a_0 x - x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} + 6x^2 \sum_{k=2}^{\infty} a_{k-2} x^{k-2}$$

$$G(x) = 5x - xG(x) + 6x^2 G(x)$$

$a_0 = 0$ and $a_1 = 5$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = -\frac{5x}{(6x^2 - x - 1)}$$

$$G(x) = -\left(\frac{5x}{(2x-1)(3x+1)}\right)$$

Counting Problems and Generating Functions
Example 4

$$G(x) = -\left(\frac{5x}{(2x-1)(3x+1)}\right)$$

$$G(x) = -\left(\frac{1}{2x-1} + \frac{1}{3x+1}\right)$$

$$G(x) = \frac{1}{1-2x} - \frac{1}{1+3x}$$

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

$$G(x) = \sum_{k=0}^{\infty} (2)^k x^k - \sum_{k=0}^{\infty} (-3)^k x^k$$

$$G(x) = \sum_{k=0}^{\infty} \left((2)^k - (-3)^k\right) x^k$$

$$a_k = \left((2)^k - (-3)^k\right)$$

Counting Problems and Generating Functions
Example 5

- Solve the recurrence relation $a_n = -2a_{n-1} - a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = -6$
- Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0 x^0 + a_1 x^1 + \sum_{k=2}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1 x + \sum_{k=2}^{\infty} (-2a_{k-1} - a_{k-2}) x^k$$

$$G(x) = a_0 + a_1 x - 2 \sum_{k=2}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} a_{k-2} x^k$$

Counting Problems and Generating Functions

Example 5

$$G(x) = a_0 + a_1 x - 2\sum_{k=2}^{\infty} a_{k-1}x^k - \sum_{k=2}^{\infty} a_{k-2}x^k$$

$$G(x) = a_0 + a_1 x + 2a_0 x - 2a_0 x - 2x\sum_{k=2}^{\infty} a_{k-1}x^{k-1} - x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = a_0 + a_1 x + 2a_0 x - 2x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} - x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = 5 + 4x - 2xG(x) - x^2G(x)$$

$a_0 = 5$ and $a_1 = -6$

$$G(x) = \frac{5+4x}{(x^2+2x+1)}$$

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

$$G(x) = \frac{5+4x}{(1+x)^2}$$

Counting Problems and Generat

Example 5

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

$$f(x)g(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(a_j b_{k-j})x^k$$

$$G(x) = \frac{5+4x}{(1+x)^2}$$

$$G(x) = \frac{5(1+x)-x}{(1+x)^2}$$

$$G(x) = \frac{5}{(1+x)} - \frac{x}{(1+x)^2}$$

$$G(x) = \sum_{k=0}^{\infty} 5(-1)^k x^k - \frac{x}{(1+x)^2}$$

$$G(x) = \sum_{k=0}^{\infty} 5(-1)^k x^k + \sum_{k=0}^{\infty} k(-1)^k x^k$$

$$G(x) = \sum_{k=0}^{\infty} (5(-1)^k + k(-1)^k)x^k$$

$$a^n = 5(-1)^n + n(-1)^n$$

$$\frac{x}{(1+x)^2} = x\frac{1}{(1+x)}\frac{1}{(1+x)}$$

$$= x\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)$$

$$= x\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^k x^k$$

$$= x\sum_{k=0}^{\infty}(-1)^k(k+1)x^k$$

$$= \sum_{k=0}^{\infty}(k+1)(-1)^k x^{k+1}$$

$$= -\sum_{k=0}^{\infty}(k+1)(-1)^{k+1}x^{k+1}$$

$$= -\sum_{k=0}^{\infty}k(-1)^k x^k$$

Counting Problems and Generating Functions

Example 6

- The sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_1 = 9$

- Use generating functions to find an explicit formula for a_n

Counting Problems and Generating Functions

Example 6

$a_n = 8a_{n-1} + 10^{n-1}$ $a_1 = 9$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) = \sum_{n=0}^{\infty}(8a_{n-1} + 10^{n-1})x^n$$

$$G(x) = a_0 + \sum_{n=1}^{\infty}(8a_{n-1} + 10^{n-1})x^n$$

$$G(x) = a_0 + \sum_{n=1}^{\infty} 8a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$G(x) = a_0 + 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=0}^{\infty} 10^{n-1}x^{n-1}$$

$$G(x) = a_0 + 8xG(x) + \frac{x}{1-10x}$$

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

Example 6

$$a_n = 8a_{n-1} + 10^{n-1} \qquad a_1 = 9$$

$$a_1 = 8a_0 + 10^{1-1} = 9$$
$$a_0 = 1$$

$$G(x) = a_0 + 8xG(x) + \frac{x}{1-10x}$$

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

$$a_n = \frac{1}{2}\left(8^n + 10^n\right)$$