Chapter 4: Advanced Counting Techniques

4.1
Recurrence Relations

4.2
Solving Linear Recurrence Relations

4.4
Generating Functions

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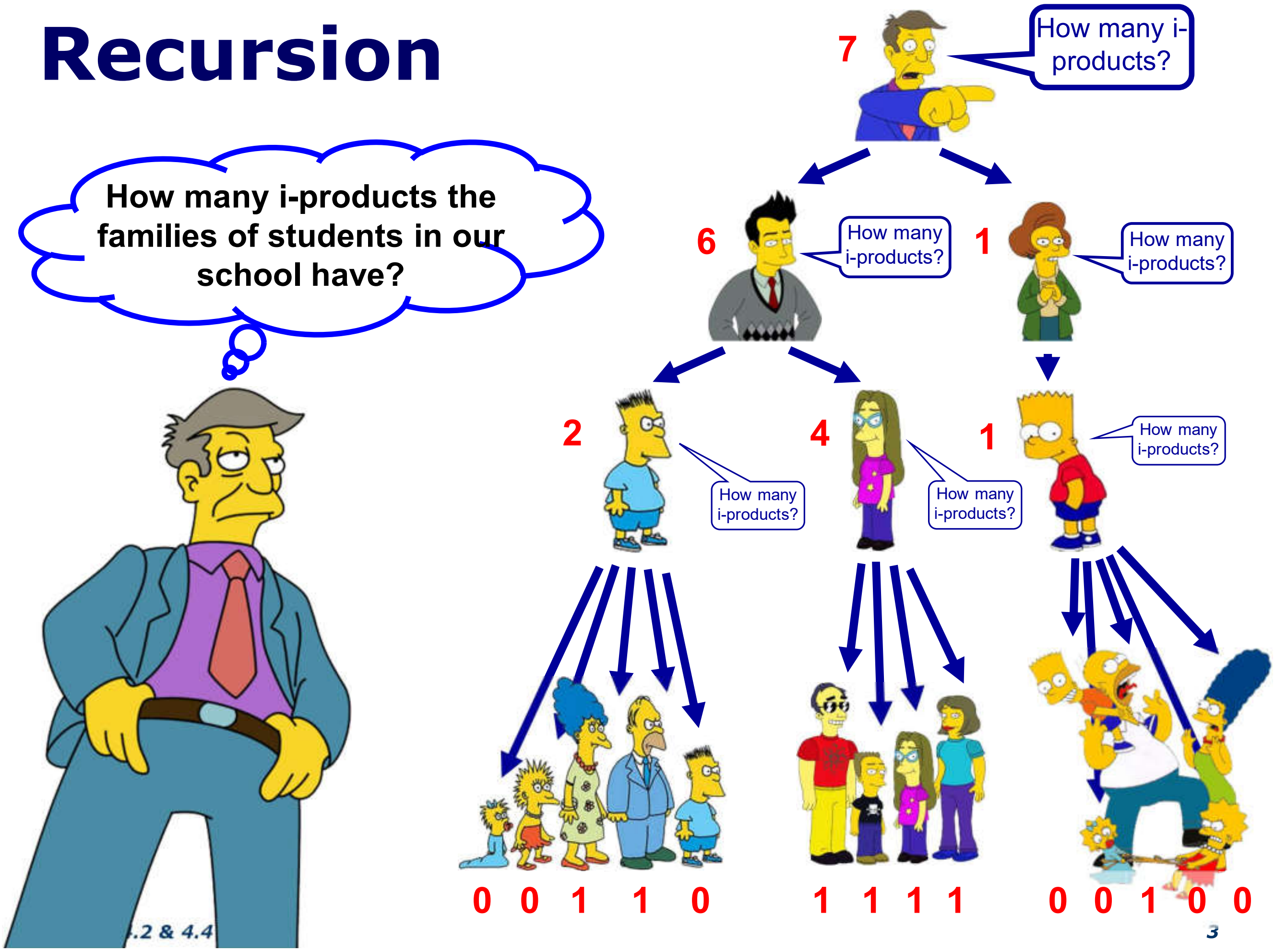
Agenda

- Recurrence Relations
- Modeling with Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations with Constant Coefficients
- Generating Functions
- Useful Facts About Power Series
- Extended Binomial Coefficient
- Extended Binomial Theorem
- Counting Problems and Generating Functions
- Using Generating Functions to Solve Recurrence Relations

Recursion



Recurrence Relations

- A **recurrence relation** for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements $(a_0, \ldots, a_{n-1})$
- A sequence is called a **solution** of a recurrence relation **if** its terms satisfy the recurrence relation

Recurrence Relations

Example 1

- Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2, 3, 4, . . . , and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

- From the recurrence relation:
  - $a_2 = a_1 – a_0 = 5 - 3 = 2$
  - $a_3 = a_2 – a_1 = 2 - 5 = -3$

Recurrence Relations

Example 2

- Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}, \text{ where } n \geq 2$$

- Which of the following are solutions?
  - $a_n = 3n$ ✓
    - $2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$
  - $a_n = 2^n$ ✗
    - $2a_{n-1} - a_{n-2} = 2(2^{n-1}) – n^{n-2} = 2^n \neq a_n$
  - $a_n = 5$ ✓
    - $2a_{n-1} - a_{n-2} = 2 \text{ x } 10 – 5 = 5 = a_n$

Recurrence Relations

- The **initial conditions** for a sequence specify the terms that precede the first term where the recurrence relation takes effect
  - For example
    $a_n = a_{n-1} + a_{n-2}$, what is the value of a_3?
    Answer depends on a_0 and a_1 (initial conditions)
    - $a_0 = 3$ and $a_1 = 5$: $a_2 = 8$, $\mathbf{a_3 = 13}$
    - $a_0 = 1$ and $a_1 = 2$: $a_2 = 3$, $\mathbf{a_3 = 5}$
- A **sequence** is determined uniquely by
  - Recurrence relation
  - Initial conditions

Modeling with Recurrence Relations

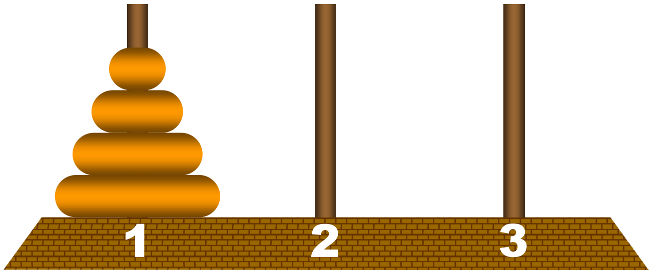
Compound Interest

- Growth of saving in a bank account with r% interest per given period
  - $\mathbf{S_n = S_{n-1} + r \cdot S_{n-1} = (r+1) \cdot S_{n-1}}$
- Example:
  - Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11 % per year with interest compounded annually. How much will be in the account after 30 years?
  - $S_{30} = 1.11S_{29} = 1.11(1.11S_{28}) = \ldots = (1.11)^{30}\ 10{,}000$

Modeling with Recurrence Relations

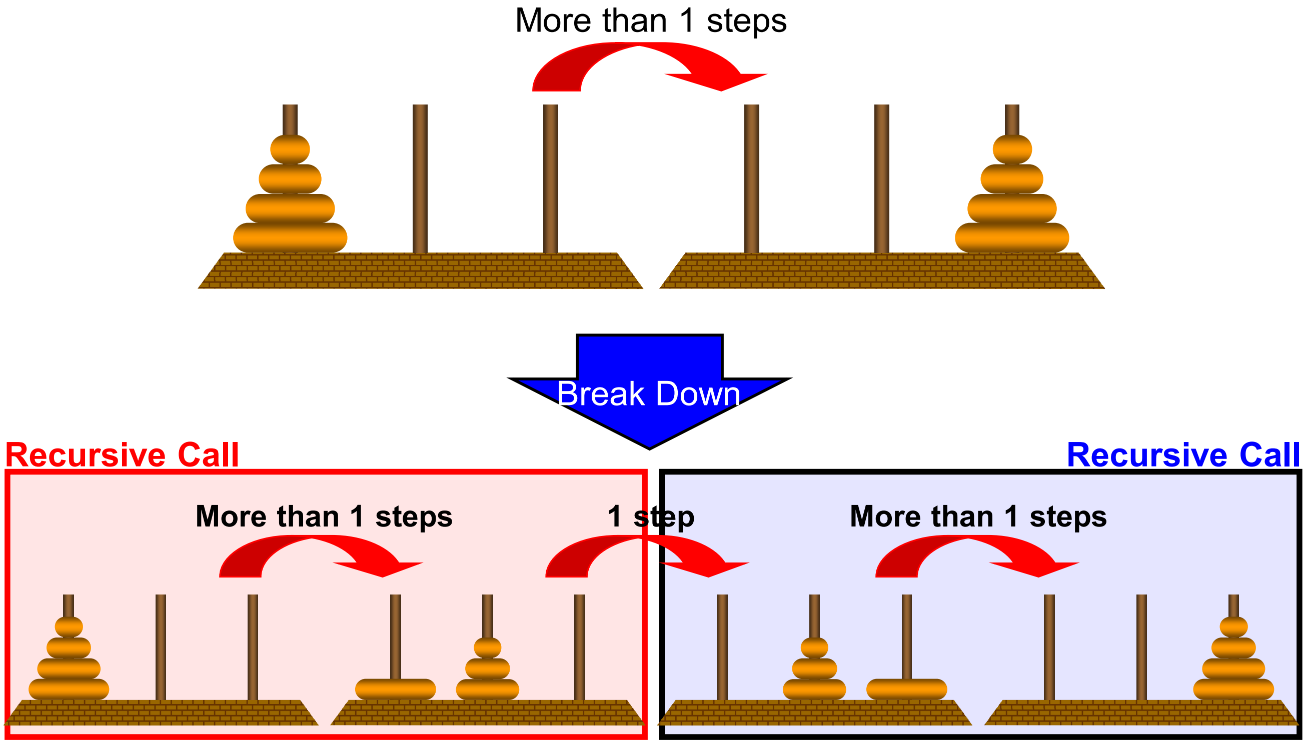
Tower of Hanoi

- **Objective**
  - Get all disks from peg 1 to peg 3
- **Rules**
  - Only move 1 disk at a time
  - Never put a larger disk on a smaller one



Modeling with Recurrence Relations

Tower of Hanoi



More than 1 steps

Break Down

Recursive Call

Recursive Call

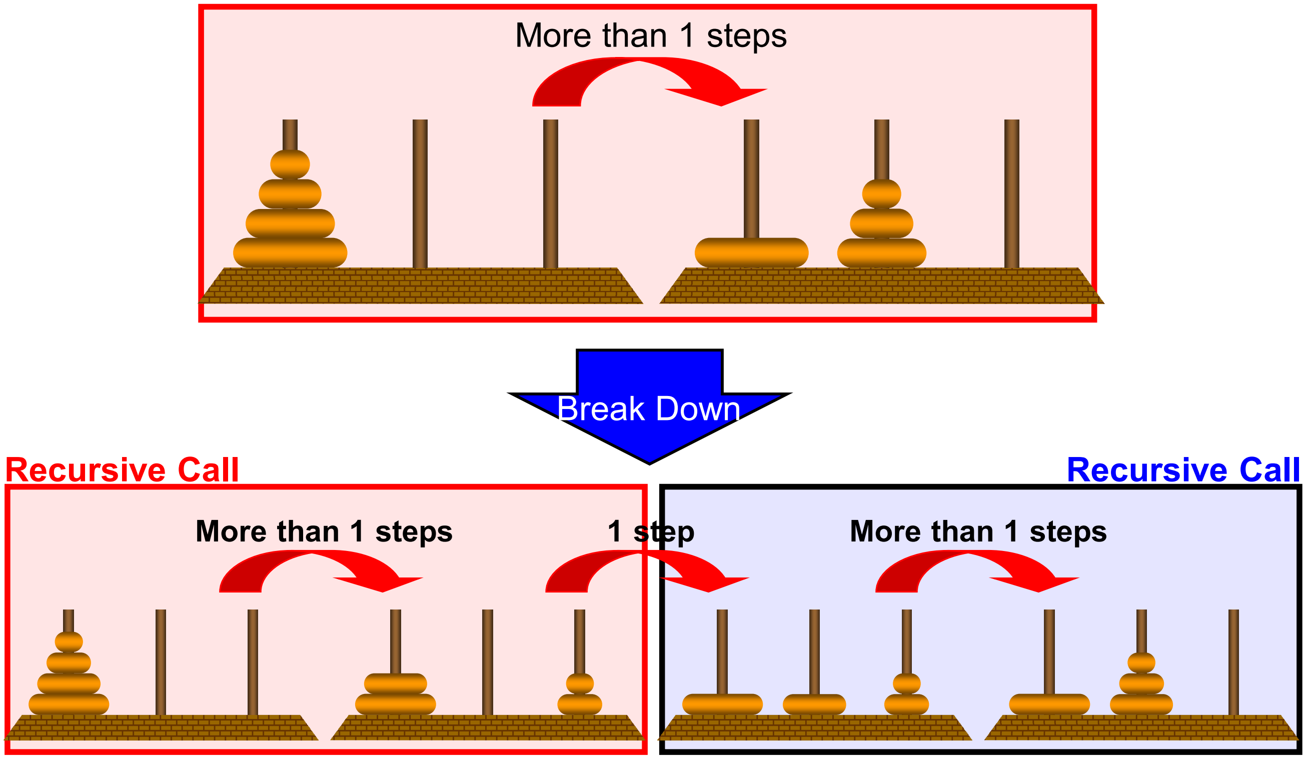
More than 1 steps

1 step

More than 1 steps

Modeling with Recurrence Relations

Tower of Hanoi



More than 1 steps

Break Down

Recursive Call

More than 1 steps

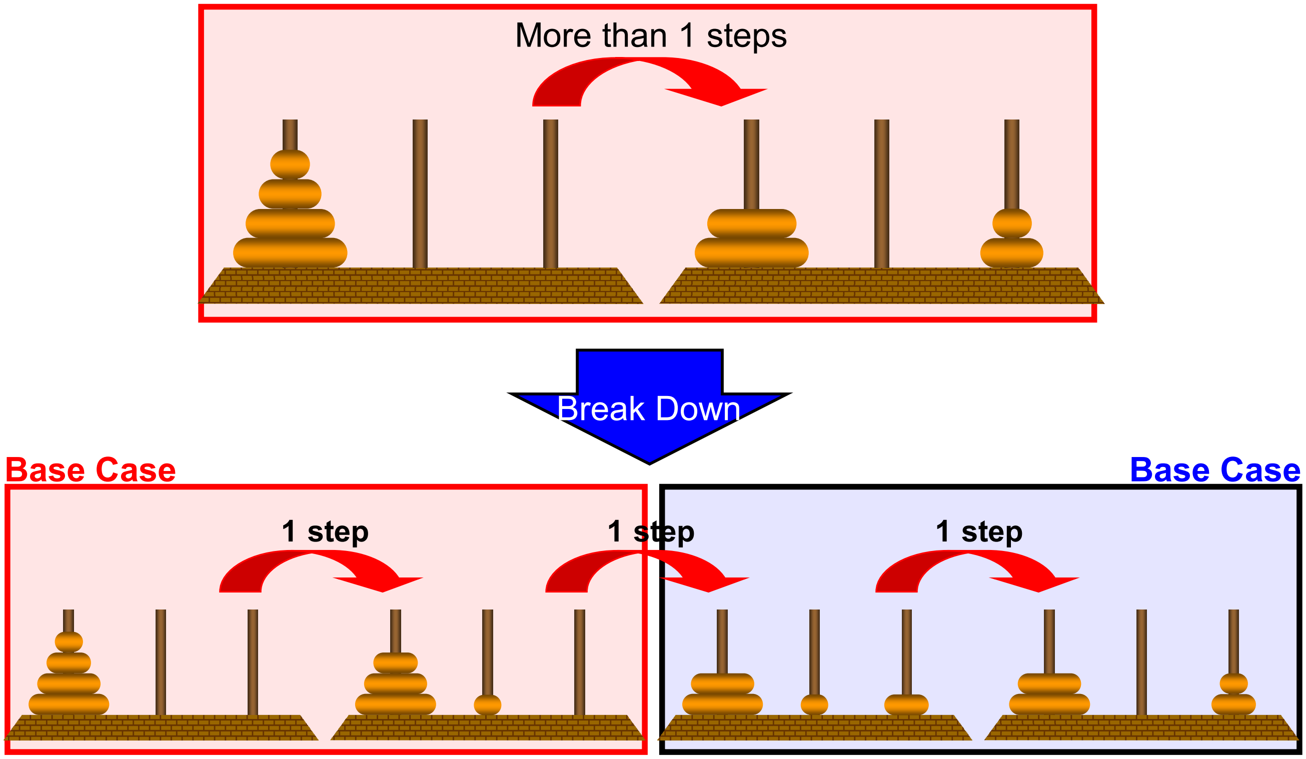
1 step

More than 1 steps

Recursive Call

Modeling with Recurrence Relations

Tower of Hanoi



More than 1 steps

Break Down

Base Case

1 step

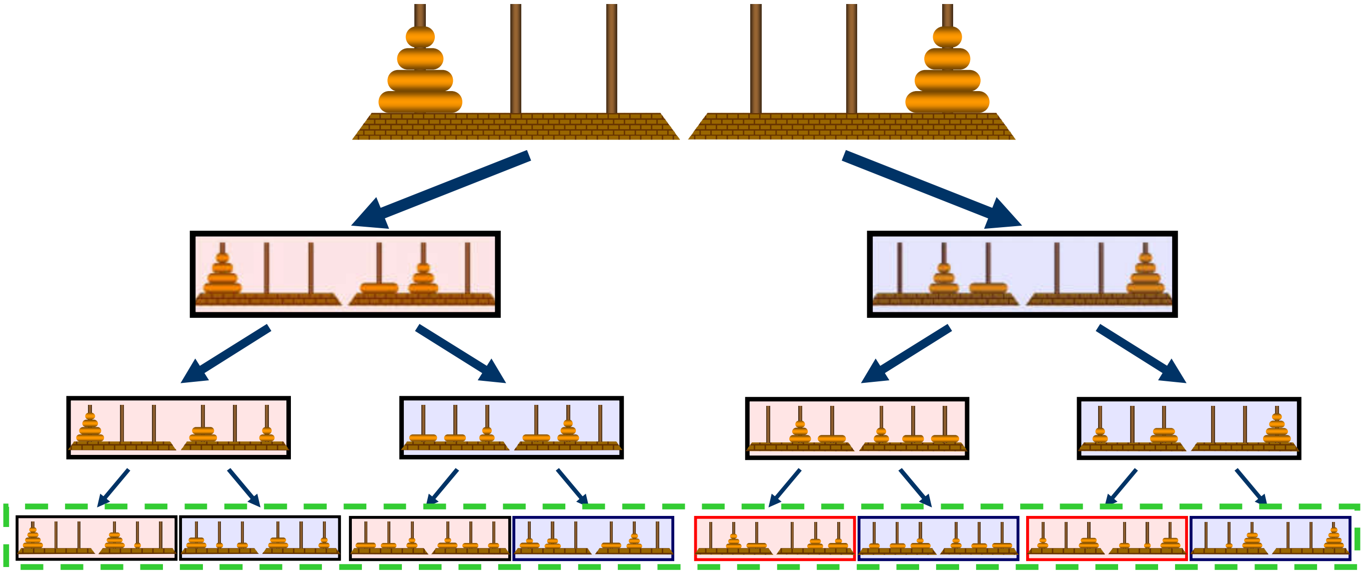
1 step

1 step

Base Case

Modeling with Recurrence Relations

Tower of Hanoi



The solution

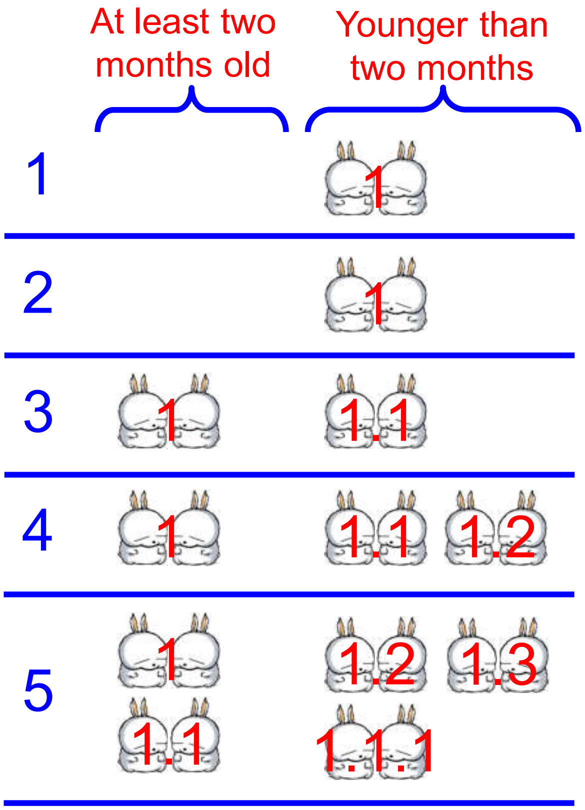
Modeling with Recurrence Relations

Tower of Hanoi

- Let H_n be the number of moves for a stack of n disks.
- Strategy:
  - Move top n−1 disks (H_{n-1} moves)
  - Move bottom disk (1 move)
  - Move top n−1 to bottom disk (H_{n-1} moves)
- $H_n = 2H_{n-1} + 1$

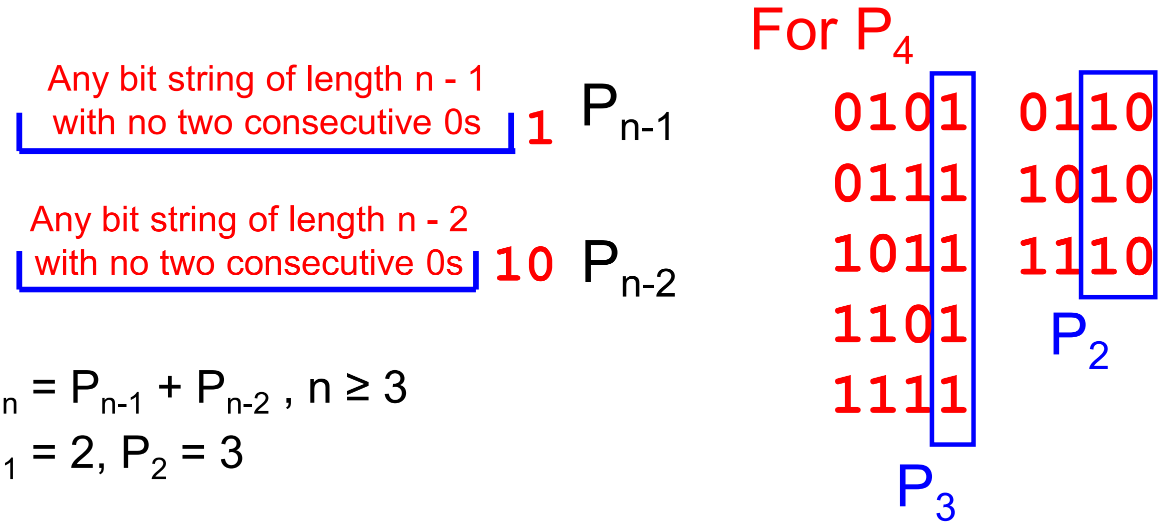
Fibonacci (Rabbits) Numbers

- A young pair of rabbits (one of each sex) is placed on an island
- A pair of rabbits does not breed until they are 2 months old
- After they are 2 months old, each pair of rabbits produces another pair each month
- $P_n = P_{n-1} + P_{n-2}$



Example 1

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s.
- Let P_n denote the number of bit strings of length n that do not have two consecutive 0s



Any bit string of length n - 1 with no two consecutive 0s 1 P_{n-1}

Any bit string of length n - 2 with no two consecutive 0s 10 P_{n-2}

For P_4

0101 0110
0111 1010
1011 1110
1101
1111

P_3 P_2

- $P_n = P_{n-1} + P_{n-2}$, $n \geq 3$
- $P_1 = 2$, $P_2 = 3$

Modeling with Recurrence Relations

Example 2

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have three consecutive 0s.

- Let P_n denote the number of bit strings of length n that do not have three consecutive 0s

$$\underbrace{\text{Any bit string of length n - 1 with no three consecutive 0s}}\ 1 \quad P_{n-1}$$

$$\underbrace{\text{Any bit string of length n - 2 with no three consecutive 0s}}\ 10 \quad P_{n-2}$$

$$\underbrace{\text{Any bit string of length n - 3 with no three consecutive 0s}}\ 100 \quad P_{n-2}$$

- $P_n = P_{n-1} + P_{n-2} + P_{n-3}$, $n \geq 4$
- $P_1 = 2, P_2 = 4, P_3 = 7$

For P_4

001$\boxed{1}$
010$\boxed{1}$
011$\boxed{1}$
100$\boxed{1}$
101$\boxed{1}$
110$\boxed{1}$
111$\boxed{1}$

P_3

00$\boxed{10}$
01$\boxed{10}$
10$\boxed{10}$
11$\boxed{10}$

P_2

0$\boxed{100}$
1$\boxed{100}$

P_1

Solving Linear Recurrence Relations

- Given $P_n = P_{n-1} + P_{n-2}$, what is P_{100}?
- It is not easy to calculate
- Need a **better solution** which is **not in relation form**
  E.g. P_n = n*10 -1

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence of Degree k with Constant Coefficients is a recurrence of the form

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} = \sum_{i=1}^{k} c_i a_{n-i}$$

where the c_i are all real numbers, and $c_k \neq 0$

  - **Li**near: the power of all a_i term is one
  - **Ho**mogeneous: no constant term (no team without a_i)
  - Recurrence: a sequence $\{a_n\}$ which a_n in terms of $a_{n-1}, a_{n-2}, \ldots$
  - Degree **k**: refer to k previous terms a_{n-k}
  - **Co**nstant **Co**efficients: c_1, c_2, … independent from n
- The short name is "**k-LiHoReCoCo**"

Solving Linear Recurrence Relations

Example

- $M_n = M_{n-1} + (1.11)M_{n-1}$
  - 1-LiHoReCoCo
- $P_n = P_{n-1} + P_{n-2}$
  - 2-LiHoReCoCo
- $a_n = a_{n-5}$
  - 5-LiHoReCoCo
- $a_n = a_{n-1} + (a_{n-2})^2$
  - Not linear
- $H_n = 2H_{n-1} + 1$
  - Not homogeneous
- $B_n = nB_{n-1}$
  - Non-constant coefficient (n is a variable)

k-LiHoReCoCo

- **Li**near: the power of all a_i term is one
- **Ho**mogeneous: no constant term (no team without a_i)
- Recurrence: a sequence $\{a_n\}$ which a_n in terms of $a_{n-1}, a_{n-2}, \ldots$
- Degree **k**: refer to k previous terms a_{n-k}
- **Co**nstant **Co**efficients: c_1, c_2, … independent from n

Solving 2-LiHoReCoCos

- Given 2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and $a_0 = c$ and $a_1 = d$

- Assume s_n and t_n be the solution, $a_n = s_n$ and $a_n = t_n$
  - $s_n = c_1 s_{n-1} + c_2 s_{n-2}$ and $t_n = c_1 t_{n-1} + c_2 t_{n-2}$
- For constants w_1 and w_2

$$\begin{aligned} \underbrace{w_1 s_n + w_2 t_n}_{a_n} &= w_1(c_1 s_{n-1} + c_2 s_{n-2}) + w_2(c_1 t_{n-1} + c_2 t_{n-2}) \\ &= c_1(\underbrace{w_1 s_{n-1} + w_2 t_{n-1}}_{a_{n-1}}) + c_2(\underbrace{w_1 t_{n-2} + w_2 t_{n-2}}_{a_{n-2}}) \end{aligned}$$

- Therefore, $a_n = w_1 s_n + w_2 t_n$ is a **solution**

Solving 2-LiHoReCoCos

- By considering 1-LiHoReCoCo, $a_n = c\ a_{n-1}$
- Obviously, the general solution is $a_n = c^n a_0$
- Therefore, the solution of the form may be $a_n = r^n$

- Substitute $a_n = r^n$ to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $r^2 - c_1 r - c_2 = 0$ or $r = 0$
  - $r = 0$ is a special case since $a_n = 0$
- $r^2 - c_1 r - c_2$ is called **characteristic equation**

Solving 2-LiHoReCoCos

- Given 2-LiHoReCoCo: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and $a_0 = c$ and $a_1 = d$
- Assume $a_n = w_1 r_1^n + w_2 r_2^n$ for r_1 and r_2 are different and some constants w_1, w_2
- We know that $r_1^2 - c_1 r_1 - c_2 = 0$ and $r_2^2 - c_1 r_2 - c_2 = 0$
- Characteristic Equation: $r^2 - c_1 r - c_2 = 0$
- Characteristic Roots: r_1 and r_2
- w_1 and w_2 can be calculated by using c and d

$$\begin{cases} a_0 = c = w_1 r_1^0 + w_2 r_2^0 \\ a_1 = d = w_1 r_1^1 + w_2 r_2^1 \end{cases}$$

Solving 2-LiHoReCoCos

Theorem

- Consider an arbitrary 2-LiHoReCoCo:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- By substituting $a_n = r^n$, we have the characteristic equation:

$$r^2 - c_1 r - c_2 = 0$$

- If there has two different roots r_1 and r_2, then

$$a_n = w_1 r_1^n + w_2 r_2^n$$

for $n \geq 0$ and some constants w_1, w_2

Solving 2-LiHoReCoCos

- **Proof**

  Given r_1, r_2 are the characteristic root

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \Leftrightarrow \quad a_n = w_1 r_1^{\,n} + w_2 r_2^{\,n}$$

  where and w_1, w_2 are constants

  - **Two steps** for the proof
    1. Show if $a_n = w_1 r_1^{\,n} + w_2 r_2^{\,n}$, $\{a_n\}$ is a solution of the recurrence relation
    2. Show if $\{a_n\}$ is the solution of the recurrence relation, $a_n = w_1 r_1^{\,n} + w_2 r_2^{\,n}$ for some w_1 and w_2

Solving 2-LiHoReCoCos

- **Step 1**
  Show if $a_n = w_1 r_1^{\,n} + w_2 r_2^{\,n}$, $\{a_n\}$ is a solution of the recurrence relation

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1\left(w_1 r_1^{n-1} + w_2 r_2^{n-1}\right) + c_2\left(w_2 r_1^{n-2} + w_2 r_2^{n-2}\right) \\ &= w_1 r_1^{n-2}\left(c_1 r_1 + c_2\right) + w_2 r_2^{n-2}\left(c_1 r_2 + c_2\right) \\ &= w_1 r_1^{n-2} r_1^2 + w_2 r_2^{n-2} r_2^2 \\ &= w_1 r_1^n + w_2 r_2^n \\ &= a_n \end{aligned}$$

r_1 and r_2 are the solution of $r^2 - c_1 r - c_2 = 0$

Solving 2-LiHoReCoCos

- **Step 2**
  Show if $\{a_n\}$ is the solution of the recurrence relation, $a_n = w_1r_1^n + w_2r_2^n$ for some w_1 and w_2
  - Suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold
  - We want to show that there are constants w_1 and w_2 such that the sequence $\{a_n\}$ with $a_n = w_1r_1^n + w_2r_2^n$ satisfies these same initial conditions
    - $a_0 = C_0 = w_1 + w_2$ and $a_1 = C_1 = w_1r_1 + w_2r_2$
  - By solving these two equations:

$$w_1 = \frac{C_1 - C_0r_2}{r_1 - r_2} \qquad w_2 = \frac{C_0r_2 - C_1}{r_1 - r_2}$$

  - When $r_1 \neq r_2$, $\{a_n\}$ with $w_1r_1^n + w_2r_2^n$ satisfy the 2 initial conditions

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Solving 2-LiHoReCoCos

- We know that $\{a_n\}$ and $\{\alpha_1r_1^n + \alpha_2r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$
- Because there is a unique solution of 2-LiHoReCoCo with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for all nonnegative integers n
- We have completed the proof

Solving 2-LiHoReCoCos
Example 1

2-LiHoReCoCo: $a_n = c_1a_{n-1} + c_2a_{n-2}$,
Characteristic Equation: $r^2 - c_1r - c_2 = 0$
$a_n = w_1r_1^n + w_2r_2^n$ (r_1 and r_2 are different)

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$
- Characteristic Equation: $r^2 - r - 2 = 0$
- Characteristic Root:
  - $r = (1 \pm 3) / 2$
  - $r = 2$ or $r = -1$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- Therefore, $a_n = w_1\, 2^n + w_2\, (-1)^n$
- By using $a_0 = 2$, $a_1 = 7$
  - $a_0 = 2 = w_1 2^0 + w_2\, (-1)^0$
  - $a_1 = 7 = w_1 2^1 + w_2\, (-1)^1$
  - $w_1 = 3$ and $w_2 = 1$
- Therefore, $a_n = 3 \cdot 2^n - (-1)^n$

Solving 2-LiHoReCoCos
Example 2

2-LiHoReCoCo: $a_n = c_1a_{n-1} + c_2a_{n-2}$,
Characteristic Equation: $r^2 - c_1r - c_2 = 0$
$a_n = w_1r_1^n + w_2r_2^n$ (r_1 and r_2 are different)

- Find an explicit formula for the Fibonacci numbers
- Recall $f_n = f_{n-1} + f_{n-2}$
  - Characteristic equation: $r^2 - r - 1 = 0$
  - Characteristic roots: $r_1 = (1 + \sqrt{5})/2$ $\quad r_2 = (1 - \sqrt{5})/2$

$$\boxed{f_n = w_1\left(\frac{1+\sqrt{5}}{2}\right)^n + w_2\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

- Initial conditions $f_0 = 0$ and $f_1 = 1$

$$f_0 = 0 = w_1 + w_2$$

$$f_1 = 1 = w_1\left(\frac{1+\sqrt{5}}{2}\right) + w_2\left(\frac{1-\sqrt{5}}{2}\right) \qquad w_1 = \frac{1}{\sqrt{5}} \qquad w_2 = -\frac{1}{\sqrt{5}}$$

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Solving 2-LiHoReCoCos with two same roots

Theorem

- Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0
- A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $\boldsymbol{a_n = w_1 r_0^n + w_2 n r_0^n}$, for $n = 0, 1, 2, \ldots$, where w_1 and w_2 are constants

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Solving 2-LiHoReCoCos with

Example 1

2-LiHoReCoCo: $\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2}}$,
Characteristic Equation: $\boldsymbol{r^2 - c_1 r - c_2 = 0}$
$\boldsymbol{a_n = w_1 r_0^n + w_2 n\, r_0^n}$

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?
- Characteristic equation: $r^2 - 6r + 9 = 0$
- Only one characteristic root: $r = 3$
- Hence, the solution to this recurrence relation is

$$\boldsymbol{a_n = w_1 3^n + w_2 n 3^n}$$

  for some constants α_1 and α_2
- By using the initial conditions,

  $a_0 = 1 = w_1$, $a_1 = 6 = 3w_1 + 3w_2$, so $w_1 = 1$ and $w_2 = 1$
- Consequently, $\boldsymbol{a^n = 3^n + n3^n}$

Solving 2-LiHoReCoCos

Summary

- Given : $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $a_0 = c$ and $a_1 = d$

  1. **Characteristic equation:** $r^2 - c_1 r - c_2 = 0$

  2a. If Characteristic Root (r_1 and r_2) are different

    1. $a_n = w_1 r_1^n + w_2 r_2^n$ is the solution
    2. Use $a_0 = w_1 + w_2 = c$ and $a_1 = w_1 r_1 + w_2 r_2 = d$ to solve w_1 and w_2

  2b. If Characteristic Root (r_1 and r_2) are the same

    1. $a_n = w_1 r^n + w_2\, n\, r^n$ is the solution
    2. Use $a_0 = w_1 = c$ and $a_1 = w_1 r_1 + w_2 r_2 = d$ to solve w_1 and w_2

☺ Small Exercise ☺

- What is the solution of the recurrence relation $a_n = - a_{n-1} + 6a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 5$?

- What is the solution of the recurrence relation $a_n = - 2a_{n-1} - a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = -6$?

☺ Small Exercise ☺

- the recurrence relation: $a_n = - a_n + 6a_{n-2}$
- Initial conditions $a_0 = 0$ and $a_1 = 5$
- Characteristic Equation: $r^2 + r - 6 = 0$
  $(r + 3)(r – 2) = 0$
- Characteristic Root: $r_1 = -3, r_2 = 2$
- Therefore, $a^n = w_1 (-3)^n + w_2 (2)^n$
- Using the initial condition
  - $a_0 = 0 = w_1 + w_2$
  - $a_1 = 5 = -3w_1 + 2w_2$
  - $w_1 = -1, w_2 = 1$
- Therefore, $a^n = - (-3)^n + (2)^n$

☺ Small Exercise ☺

- the recurrence relation: $a_n = - 2a_n - a_{n-2}$
- Initial conditions $a_0 = 5$ and $a_1 = -6$
- Characteristic Equation: $r^2 + 2r + 1 = 0$
  $(r + 1)(r + 1) = 0$
- Characteristic Root: $r_1 = -1$
- Therefore, $a^n = w_1 (-1)^n + w_2 n (-1)^n$
- Using the initial condition
  - $a_0 = 5 = w_1$
  - $a_1 = -6 = -w_1 - w_2$
  - $w_1 = 5, w_2 = 1$
- Therefore, $a^n = 5 (-1)^n + n (-1)^n$

Solving k-LiHoReCoCos

- k-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$

2-kiHoReCoCo

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2}}$$

- **Characteristic Equation** is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$

$$\boxed{r^2 - c_1 r - c_2 = 0}$$

- **Theorem**
  If there are k distinct roots r_i , then the solutions to the recurrence are of the form:

$$a_n = \sum_{i=1}^{k} w_i r_i^n$$

$$\boxed{a_n = w_1 r_1{}^n + w_2 r_2{}^n}$$

  for all $n \geq 0$, where the w_i are constants

Solving k-LiHoReCoCos

Example

- Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

  with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

- The characteristic equation is:

$$r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$$

  - The characteristic roots are $r = 1$, $r = 2$, and $r = 3$
- $a_n = w_1 1^n + w_2 2^n + w_3 3^n$
- By using the initial conditions
  - $a_0 = 2 = w_1 + w_2 + w_3$
  - $a_1 = 5 = w_1 + w_2 \text{ x } 2 + w_3 \text{ x } 3$
  - $a_2 = 15 = w_1 + w_2 \text{ x } 4 + w_3 \text{ x } 9$
  - Therefore, $w_1 = 1$, $w_2 = -1$ and $w_3 = 2$
- As a result, $a_n = 1 - 2^n + 2 \text{ x } 3^n$

Solving k-LiHoReCoCos with same roots

- Let $c_1, c_2, \ldots, c_k$ be real numbers
- Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$

  - i.e. r_i appear m_i times
  - $m_1 + m_2 + \cdots + m_t = k$

Solving k-LiHoReCoCos with same roots

Special case for k=2, One distinct root

$$a_n = w_1 r_0^{\,n} + w_2 n r_0^{\,n}$$

$$a_n = (w_1 + w_2 n) r_0^{\,n}$$

- A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

- If and only if

$$\left.\begin{aligned} a_n &= \overbrace{(w_{1,0} + w_{1,1} n + \ldots + w_{1,m_1-1} n^{m_1-1})}^{\text{Multiplicities for } r_i} r_1^n \\ &\quad + (w_{2,0} + w_{2,1} n + \ldots + w_{2,m_2-1} n^{m_2-1}) r_2^n \\ &\quad + \ldots + (w_{t,0} + w_{t,1} n + \ldots + w_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}\right\} \text{No. of distinct roots}$$

$$= \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i-1} w_{i,j} n^j \right) r_i^n$$

for $n = 0,1,2, \ldots$, where $w_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_{i\text{-}1}$

Solving k-LiHoReCoCos with s

$$\begin{aligned} a_n = &(w_{1,0} + w_{1,1}n + \ldots + w_{1,m_1-1}n^{m_1-1})r_1^n \\ &+(w_{2,0} + w_{2,1}n + \ldots + w_{2,m_2-1}n^{m_2-1})r_2^n \\ &+\ldots+(w_{t,0} + w_{t,1}n + \ldots + w_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

Example

- Find the solution to the recurrence relation

$$H_n = -H_{n-1} + 3H_{n-2} + 5H_{n-3} + 2H_{n-4}$$

with the initial conditions $H_0 = 1, H_1 = 0, H_2 = 1, H_3 = 2$

- **Characteristic equation**: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

$$(x-2)(x+1)^3 = 0$$

- **Roots**: $-1, -1, -1, 2$
- Therefore: $H_n = (c_1 + c_2 n + c_3 n^2)(-1)^n + c_4 2^n$
- By initial conditions:

$$\begin{cases} H_0 = c_1 + c_4 = 1 \\ H_1 = -c_1 - c_2 - c_3 + 2c_4 = 0 \\ H_2 = c_1 + 2c_2 + 4c_3 + 4c_4 = 1 \\ H_3 = -c_1 - 3c_2 - 9c_3 + 8c_4 = 2 \end{cases}$$

$$c_1 = \frac{7}{9}, c_2 = -\frac{1}{3}, c_3 = 0, c_4 = \frac{2}{9}$$

$$H_n = \frac{7}{9}(-1)^n - \frac{1}{3}n(-1)^n + \frac{2}{9}2^n$$

Solving k-LiHoReCoCos

Summary

- Given : $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ and $a_i = c_i$, where $i = 1, 2, \ldots, k$

1. Characteristic equation: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$

2. Characteristic Root (r_1, r_2, …, r_k)

   1. $a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1} w_{i,j} n^j\right) r_i^n$ is the solution of k-LiHoReCoCos

      where m_i is the multiplicity of r_i

   2. solve w_i by $a_p = c_p = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1} w_{i,j} p^j\right) r_i^p$

      where p = 1, 2, ..., k

☺ Small Exercise ☺

- What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 0$, $a_1 = 8$ and $a_2 = 4$?

$$\boxed{\begin{aligned} a_n = {} & (w_{1,0} + w_{1,1}n + \ldots + w_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (w_{2,0} + w_{2,1}n + \ldots + w_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \ldots + (w_{t,0} + w_{t,1}n + \ldots + w_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}}$$

☺ Small Exercise ☺

- the recurrence relation: $a_n = a_{n-1} + a_{n-2} - a_{n-3}$
- Initial conditions $a_0 = 0$, $a_1 = 8$ and $a_2 = 4$
- Characteristic Equation: $r^3 - r^2 - r + 1 = 0$
  $(r - 1)(r - 1)(r + 1) = 0$
- Characteristic Root: $r_1 = 1, r_2 = 1, r_3 = -1$
- Therefore, $a^n = (c_1 + c_2\, n)\ (1)^n + c_3\ (-1)^n$
- Using the initial condition
  - $a_0 = 0 = c_1 + c_3$
  - $a_1 = 8 = c_1 + c_2 - c_3$
  - $a_2 = 4 = c_1 + 2c_2 + c_3$
  - $c_1 = 3, c_2 = -3, c_3 = 2$
- Therefore, $a^n = 3\ (1 - n)\ (1)^n + 2\ (-1)^n$

Solving LiNoReCoCos

- Linear nonhomogeneous recurrence of degree k with constant coefficients (k-LiNoReCoCos) contain some terms $F(n)$ that depend only on n but not a_i
- **General form**:

$$\underbrace{a_n = c_1 a^{n-1} + \ldots + c_k a^{n-k}}_{\text{Associated Homogeneous Recurrence Relation}} + F(n)$$

Solving LiNoReCoCos

- If { $a_n^{(p)}$ } is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n\text{-}1} + c_2 a_{n\text{-}2} + \ldots + c_k a_{n\text{-}k} + F(n)$$

- Then every solution is of the form { $a_n^{(p)} + a_n^{(h)}$ }, where { $a_n^{(h)}$ } is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n\text{-}1} + c_2 a_{n\text{-}2} + \ldots + c_k a_{n\text{-}k}$$

Solving LiNoReCoCos

- **Proof**
  - As $\{a_n^{(p)}\}$ is a particular solution for LiNoReCoCos
  - Suppose that $\{b_n\}$ is an another solution

$$\begin{array}{rrl} & a_n^{(p)} &= c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n) \\ - & b_n &= c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n) \\ \hline & b_n - a_n^{(p)} &= c_1\left(b_{n-1} - a_{n-1}^{(p)}\right) + c_2\left(b_{n-2} - a_{n-2}^{(p)}\right) + \ldots + c_k\left(b_{n-k} - a_{n-k}^{(p)}\right) \\ & a_n^{(h)} &= c_1 a_{n-1}^{(h)} + c_2 a_{n-2}^{(h)} + \ldots + c_k a_{n-k}^{(h)} \end{array}$$

  - $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, named $\{a_n^{(h)}\}$
  - Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Solving LiNoReCoCos

Example 1

$$\boxed{\begin{array}{l} a_n = 3a_{n-1} + 2n \\ a_1 = 3 \end{array}}$$

- Find all solutions to $a_n = 3a_{n-1} + 2n$, which solution has $a_1 = 3$?

- Notice this is a 1-LiNoReCoCo.

- Its associated 1-LiHoReCoCo
  - $a_n = 3a_{n-1}$ and root is 3
  - Solution is $a_n^{(h)} = c3^n$

- The solutions of LiNoReCoCo are in the form

$$a_n = \boxed{a_n^{(p)}} + a_n^{(h)}$$

Next step

Solving LiNoReCoCos
Example

$a_n = 3a_{n-1} + 2n$
$a_1 = 3$

- If $F(n)$ is a degree-u polynomial in n, a degree-u polynomial should be tried as the particular solution $a_n^{(p)}$
- Now, $F(n) = 2n$
  - Try $a_n^{(p)} = cn + d$, c and d are constants

$$a_n = 3a_{n\text{-}1} + 2n$$

$$cn+d = 3(c(n-1)+d) + 2n$$

$$(2c+2)n + (3c-2d) = 0$$

$$c = -1 \text{ and } d = -3/2$$

  - Solution is: $a_n^{(p)} = \text{-}n - 3/2$

Solving LiNoReCoCos
Example

$a_n = 3a_{n-1} + 2n$
$a_1 = 3$

- Therefore, we have

$$a_n = a_n^{(p)} + a_n^{(h)}$$
$$= -n - \frac{3}{2} + c3^n$$

$$a_n^{(p)} = -n - \frac{3}{2}$$

$$a_n^{(h)} = c3^n$$

- By using $a_1 = 3$
  - $3 = \text{-}1 \text{ - } 3/2 + 3c$
  - $c = 11/6$
- As a result, $a_n = -n - \frac{3}{2} + \frac{11 \cdot 3^n}{6}$

Solving LiNoReCoCos

Particular Solution

- Suppose $\{a_n\}$ satisfies the LiNoReCoCo $a_n = \left(\sum_{i=1}^{k} c_i a_{n-i}\right) + F(n)$

  where c_i $(i = 1,2,\ldots,k)$ are real numbers and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0)\boxed{s^n}$$

  where $b_0, b_1, \ldots, b_t$ and s are real numbers

- When **s is not a root** of the characteristic equation of the associated linear homogeneous RR, there is a particular solution ($a_n^{(p)}$) of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$$

- When **s is a root** of the characteristic equation of the associated linear homogeneous RR with multiplicity m, there is a particular solution ($a_n^{(p)}$) of the form

$$\boxed{n^m}(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$$

Solving LiNoRe

Example

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0)s^n$$

s is not a root $(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$

s is a root $n^m(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$

- What form d
  nonhomogen
  $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when ...

- Consider the associated homogeneous RR:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

- Characteristic Equation

$$r^2 - 6r + 9 = (r-3)^2 = 0$$

- Characteristic Root is 3, of multiplicity m=2

- $F(n) = 3^n$

$$n^2(p_0)3^n$$

- $F(n) = n3^n$

$$n^2(p_1 n + p_0)3^n$$

- $F(n) = n^2 2^n$

$$(p_2 n^2 + p_1 n + p_0)2^n$$

- $F(n) = (n^2 + 1)3^n$

$$n^2(p_2 n^2 + p_1 n + p_0)3^n$$

Solving LiNoReCoCos : Pa

Example 2

$F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0) s^n$

s is not a root $(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n$

s is a root $n^m (p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n$

- Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k$$

- a_n satisfies the linear nonhomogeneous RR

$$a_n = a_{n\text{-}1} + n$$

- Associated linear homogeneous RR is $a_n = a_{n\text{-}1}$
- Root is 1. The solution is $a_n^{(h)} = c(1)^n$, c is a constant
- Since $F(n) = n = n \times (1)^n$, and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous RR
- So the particular solution has the form $n(p_1 n + p_0)$

Solving LiNoReCoCos : Particular Solu

Example 2

$$a_n = a_{n\text{-}1} + n$$
$$a_n^{(p)} = n(p_1 n + p_0)$$

- By solving

$$p_1 n^2 + p_0 n = p_1 (n - 1)^2 + p_0 (n - 1) + n$$

- We have $p_0 = p_1 = 1/2$
- Recall, $a_n = a_n^{(p)} + a_n^{(h)}$

$$a_n = n(n+1)/2 + c$$

$$a_n^{(h)} = c$$
$$a_n^{(p)} = n(n+1)/2$$

- By using $a_1 = 1$, so $c = 0$
- Therefore,

$$a_n = \frac{n(n+1)}{2}$$

Solving Linear Recurrence Relations

Summary

- **k-LiHoReCoCos with m same roots (*without F(x)*)**
  - Find the root of characteristic equation
  - $$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$$
    $$+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$
    $$+ \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$
  - Use initial terms to find alphas
- **k-LiNoReCoCos with m same roots (*with F(x)*)**
  - Find the solution of characteristic equation of Associated linear homogeneous RR $a_n^{(h)}$
  - Find the particular solution of LiNoReCoCo using
    $$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$
  - Finally $a_n^{(p)} + a_n^{(h)}$
  - Use initial terms to find alphas

☺ Small Exercise ☺

- Find all solutions to $a_n = 7a_{n-1} + (2n^2+2)3^n$, which solution has $a_1 = 10$?

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Its associated 1-LiHoReCoCo
  - $a_n = 7a_{n-1}$ and root is 7
  - Solution is $a_n^{(h)} = c7^n$
- The solutions of 1-LiNoReCoCo are in the form

$$a_n = a_n^{(p)} + a_n^{(h)}$$

- Need to do is **find one** $a_n^{(p)}$

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Now, $F(n) = (2n^2+2)3^n$
  - $a_n^{(p)} = (an^2 + bn + c)3^n$

$$a_n = 7a_{n-1} + (2n^2+2)3^n$$

$$(an^2+bn+c)3^n = 7(a(n-1)^2+b(n-1)+c)3^{n-1} + (2n^2+2)3^n$$

$$3an^2+3bn+3c = 7an^2 - 14an + 7a + 7bn - 7b + 7c + 6n^2 + 6$$

$$0 = 4an^2 - 14an + 7a + 4bn - 7b + 4c + 6n^2 + 6$$

$$0 = n^2(4a+6)+n(4b-14a) +(4c +7a-7b+6)$$

$4a+6=0$ $4b-14a=0$ $4c +7a-7b+6=0$

$a=-3/2$ $b=-21/4$ $c=-129/16$

$$\boxed{a_n^{(p)} = (-3n^2/2-21n/4-129/6)3^n}$$

☺ Small Exercise ☺

$a_n = 7a_{n-1}+(2n^2+2)3^n$
$a_0 = 10$

- Therefore, we have

$$a_n = a_n^{(p)} + a_n^{(h)}$$

$$a_n^{(p)} = (-3n^2/2 - 21n/4 - 129/6)3^n$$

$$= (-3n^2/2 - 21n/4 - 129/6)3^n + c7^n$$

$$a_n^{(h)} = c7^n$$

- By using $a_0 = 10$
  - $a_0 = 10 = -129/6 + c$
  - $c = 189/6$
- As a result,

$$a_n = (-3n^2/2 - 21n/4 - 129/6)3^n + 189 \cdot 7^n/6$$

Generating Functions

- **Generating functions** (**G(x)**) are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series
- **Generating function** for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0x^0 + a_1x^1 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

Example

- What is the generating function for the following sequence?
  - **{0, 2, …, 2k,…}**

$$0 + 2x + \ldots + 2k \cdot x^k + \ldots = \sum_{k=0}^{\infty} 2^k x^k$$

  - **{1, 1, 1, 1, 1}**

$$1 + x + x^2 + x^3 + x^4 = \sum_{k=0}^{4} x^k$$

Useful Facts About Power Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

- $f(x) = \frac{1}{1-x}$

  is generating function of the sequence 1,1,1,1, ...

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

- $f(x) = \frac{1}{1-ax}$

  is generating function of the sequence 1, a, a^2, a^3, ...

Useful Facts About Power Series

- Given: $f(x)=\sum_{k=0}^{\infty} a_k x^k \qquad g(x)=\sum_{k=0}^{\infty} b_k x^k$

$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$$

$$f(x)g(x)=\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k$$

$$=(a_0x^0+a_1x^1+\ldots)(b_0x^0+b_1x^1+\ldots)$$

$$= x^0(a_0b_0)+x^1(a_0b_1+a_1b_0)+$$
$$x^2(a_0b_2+a_1b_1+a_2b_0)+\ldots$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(a_j b_{k-j}\right)x^k$$

Useful Facts About Power Series

Example

- Let $h(x)=\dfrac{1}{(1-x)^2}$,
- Find the coefficients a_0, a_1, a_2, ... in the expansion $h(x)=\sum_{k=0}^{\infty} a_k x^k$

$$f(x)=\frac{1}{1-x}=\sum_{k=0}^{\infty} x^k$$

$$f(x)=\sum_{k=0}^{\infty} a_k x^k \quad g(x)=\sum_{k=0}^{\infty} b_k x^k$$

$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$$

$$f(x)g(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(a_j b_{k-j}\right)x^k$$

$$h(x)=\frac{1}{(1-x)^2}=\frac{1}{(1-x)}\frac{1}{(1-x)}$$

$$=\left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right)$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{k} x^k$$

$$=\sum_{k=0}^{\infty}(k+1)x^k$$

$a_k=k+1$

Counting Problems and Generating Functions

- How can we solve the counting problems, including the recurrence relation, by using the Generating Functions?

$$G(x) = a_0 x^0 + a_1 x^1 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Counting Problems and Generating Functions

Example 1

- Find the number of solutions of $e_1 + e_2 + e_3 = n$ when $n = 17$, where e_1, e_2, e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, $4 \le e_3 \le 7$
- By considering this generating function for the sequence $\{a_n\}$, where a_n is the number of solution for n

$$\sum_{k=0}^{\infty} a_k x^k = \begin{pmatrix} (x^2 + x^3 + x^4 + x^5) \cdot \\ (x^3 + x^4 + x^5 + x^6) \cdot \\ (x^4 + x^5 + x^6 + x^7) \end{pmatrix}$$

- As $n = 17$, a_{17}, which is the coefficient of x^{17}, is the solution
- Answer is 3

Counting Problems and Generating Functions

Example 2

- In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

- By considering this generating function for the sequence $\{a_n\}$, where a_n is the number of solution for n

$$\sum_{k=0}^{\infty} a_k x^k = (x^2 + x^3 + x^4)(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)$$

- The coefficient of x^8 is 6

Counting Problems and Generating Functions

Example 3

- Solve the recurrence relation $\boldsymbol{a_k = 3a_{k-1}}$ for $k=1, 2, 3, \ldots$ and initial condition $\boldsymbol{a_0=2}$

- Let $\boldsymbol{G(x)}$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = \sum_{k=0}^{\infty} 3a_{k-1} x^k$$

$$G(x) = a_0 + \sum_{k=1}^{\infty} 3a_{k-1} x^k$$

$$G(x) = a_0 + 3x \sum_{k=1}^{\infty} a_{k-1} x^{k-1}$$

$$G(x) = a_0 + 3xG(x)$$

$$G(x) = \frac{2}{1-3x}$$

$$G(x) = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}}$$

$$a_k = 2 \cdot 3^k$$

Counting Problems and Generating Functions

Example 4

- Solve the recurrence relation $a_k = -a_{k\text{-}1} + 6a_{k\text{-}2}$ with initial conditions $a_0 = 0$ and $a_1 = 5$
- Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0x^0 + a_1x^1 + \sum_{k=2}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1x + \sum_{k=2}^{\infty} (-ak_{-1} + 6a_{k-2})x^k$$

$$G(x) = a_0 + a_1x - \sum_{k=2}^{\infty} a_{k-1}x^k + 6\sum_{k=2}^{\infty} a_{k-2}x^k$$

Counting Problems and Generating Functions

Example 4

$$G(x) = a_0 + a_1x - \sum_{k=2}^{\infty} a_{k-1}x^k + 6\sum_{k=2}^{\infty} a_{k-2}x^k$$

$$G(x) = a_0 + a_1x + a_0x - a_0x - x\sum_{k=2}^{\infty} a_{k-1}x^{k-1} + 6x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = a_0 + a_1x + a_0x - x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} + 6x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = 5x - xG(x) + 6x^2G(x) \qquad \boxed{a_0 = 0 \text{ and } a_1 = 5} \qquad \boxed{G(x) = \sum_{k=0}^{\infty} a_k x^k}$$

$$G(x) = -\frac{5x}{(6x^2 - x - 1)}$$

$$G(x) = -\left(\frac{5x}{(2x-1)\,(3x+1)}\right)$$

Counting Problems and Generating Functions

Example 4

$$G(x) = -\left(\frac{5x}{(2x-1)\,(3x+1)}\right)$$

$$G(x) = -\left(\frac{1}{2x-1} + \frac{1}{3x+1}\right)$$

$$G(x) = \frac{1}{1-2\mathrm{x}} - \frac{1}{1+3\mathrm{x}}$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}}$$

$$G(x) = \sum_{k=0}^{\infty} (2)^k x^k - \sum_{k=0}^{\infty} (-3)^k x^k$$

$$G(x) = \sum_{k=0}^{\infty} \left((2)^k - (-3)^k\right) x^k$$

$$a_k = \left((2)^k - (-3)^k\right)$$

Counting Problems and Generating Functions

Example 5

- Solve the recurrence relation $a_n = -2a_{n\text{-}1} - a_{n\text{-}2}$ with initial conditions $a_0 = 5$ and $a_1 = \text{-}6$
- Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0 x^0 + a_1 x^1 + \sum_{k=2}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1 x + \sum_{k=2}^{\infty} (-2a_{k-1} - a_{k-2}) x^k$$

$$G(x) = a_0 + a_1 x - 2\sum_{k=2}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} a_{k-2} x^k$$

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Example 5

$$G(x) = a_0 + a_1x - 2\sum_{k=2}^{\infty} a_{k-1}x^k - \sum_{k=2}^{\infty} a_{k-2}x^k$$

$$G(x) = a_0 + a_1x + 2a_0x - 2a_0x - 2x\sum_{k=2}^{\infty} a_{k-1}x^{k-1} - x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = a_0 + a_1x + 2a_0x - 2x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} - x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2}$$

$$G(x) = 5 + 4x - 2xG(x) - x^2G(x)$$

$$\boxed{a_0 = 5 \text{ and } a_1 = \text{-}6}$$

$$G(x) = \frac{5 + 4x}{(x^2 + 2x + 1)}$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1 - ax}}$$

$$G(x) = \frac{5 + 4x}{(1 + x)^2}$$

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Example 5

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1 - ax}}$$

$$\boxed{f(x)g(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(a_j b_{k-j})x^k}$$

$$G(x) = \frac{5 + 4x}{(1 + x)^2}$$

$$G(x) = \frac{5(1 + x) - x}{(1 + x)^2}$$

$$G(x) = \frac{5}{(1 + x)} - \frac{x}{(1 + x)^2}$$

$$G(x) = \sum_{k=0}^{\infty} 5(-1)^k x^k - \frac{x}{(1 + x)^2}$$

$$G(x) = \sum_{k=0}^{\infty} 5(-1)^k x^k + \sum_{k=0}^{\infty} k(-1)^k x^k$$

$$G(x) = \sum_{k=0}^{\infty}\left(5(-1)^k + k(-1)^k\right)x^k$$

$$a^n = 5\ (\text{-}1)^n + n\ (\text{-}1)^n$$

$$\frac{x}{(1 + x)^2} = x\frac{1}{(1 + x)}\frac{1}{(1 + x)}$$

$$= x\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)$$

$$= x\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^k x^k$$

$$= x\sum_{k=0}^{\infty}(-1)^k(k + 1)x^k$$

$$= \sum_{k=0}^{\infty}(k + 1)(-1)^k x^{k+1}$$

$$= -\sum_{k=0}^{\infty}(k + 1)(-1)^{k+1}x^{k+1}$$

$$= -\sum_{k=0}^{\infty}k(-1)^k x^k$$

Example 6

- The sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_1 = 9$

- Use generating functions to find an explicit formula for a_n

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Example 6

$$\boxed{a_n = 8a_{n-1} + 10^{n-1} \qquad a_1 = 9}$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) = \sum_{n=0}^{\infty} (8a_{n-1} + 10^{n-1}) x^n$$

$$G(x) = a_0 + \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1}) x^n$$

$$G(x) = a_0 + \sum_{n=1}^{\infty} 8a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$$

$$G(x) = a_0 + 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=0}^{\infty} 10^{n-1} x^{n-1}$$

$$G(x) = a_0 + 8xG(x) + \frac{x}{1 - 10x}$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1 - ax}}$$

Example 6

$$\boxed{a_n = 8a_{n-1} + 10^{n-1} \qquad a_1 = 9}$$

$$a_1 = 8a_0 + 10^{1-1} = 9$$
$$a_0 = 1$$

$$G(x) = a_0 + 8xG(x) + \frac{x}{1-10x}$$

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$\boxed{\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}}$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

$$a_n = \frac{1}{2}\left(8^n + 10^n\right)$$