Chapter 3: Counting

3.1 The Basics of Counting

3.2 The Pigeonhole Principle

3.3 Permutations & Combinations

3.5 Generalized Permutations & Combinations

3.6 Generating Permutations & Combinations

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Agenda

- Basic Counting Principles
  - Multiplication / Addition Principle
  - Inclusion-Exclusion Principle
  - Permutation / Combination
- Distributing Objects into Boxes
- Generating Permutations & Combinations

Why Counting?

- The **brute force attack** is the most common way (time consumed but effective) in hacking
- How security of your password?
  - 5 digits at most
  - Each digit either 0-9, a-z or A-Z
- How many times a hacker need to try in the worst situation?

Why Counting?

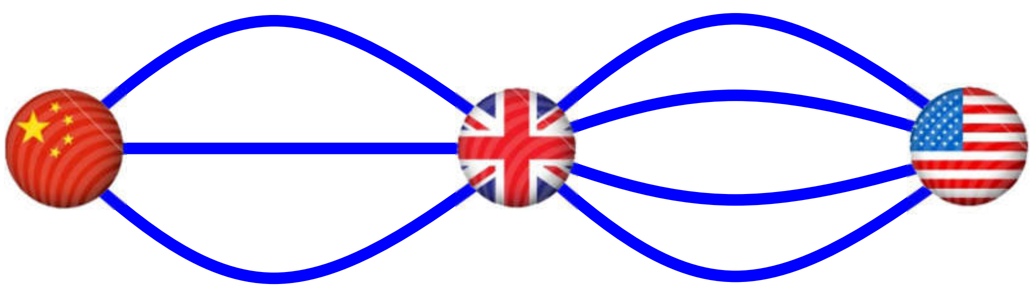
- Counting problems arise throughout mathematics and computer science
  - For example
    - the number of experiment outcomes
    - the number of operations in an algorithm (time complexity)

Basic Counting Principle

- Multiplication / Addition Principle
- Inclusion-Exclusion Principle
- Permutation / Combination

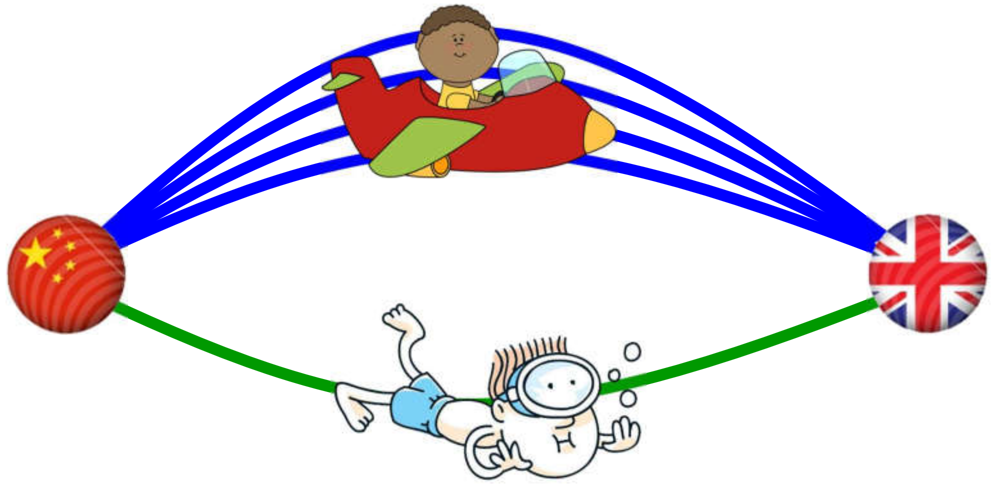
Multiplication (Product) Rule

- If a task can be constructed in t successive steps and step i can be done in n_i ways, where i = 1…t, then the number of different possible ways is $n_1 \times n_2 \times ... \times n_m$



Addition (Sum) Rule

- If a task can be done in one of n_1 ways, in one of n_2 ways, ... , or in one of n_m ways, where all sets of n_j ways are disjoint, then the number of ways is $n_1 + n_2 + \ldots + n_m$



Example 1

- In 1999, a virus named "**Melissa**" is created by David L. Smith based on a Microsoft Word macro
- Melissa sends an email "Here is that document you asked for, don't show it to anybody else." to the top 50 people in the address book

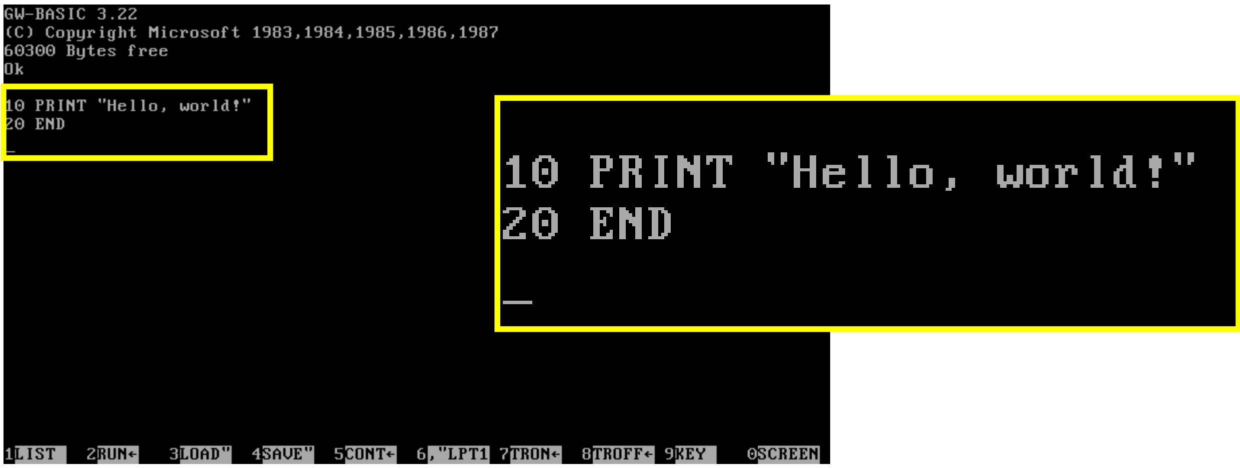
- How many emails are sent after 4 iterations?
  - 1st iteration: **1**
  - 2nd iteration: **1 x 50 = 50**
  - 3rd iteration: **50 x 50 = 2,500**
  - 4th iteration: **2500 x 50 = 6,250,000**

(By Multiplication Rule)

6,377,551
(By Addition Rule)

Example 2

- A programming language Beginner's All-purpose Symbolic Instruction Code (**BASIC**)
- GW-BASIC (1986) in MS-DOS



Example 2

- In **BASIC**, the requirements of a variable name
  - A string of 1 or 2 alphanumeric characters (a-z or 0-9)
  - Begin with a letter
  - Uppercase and lowercase letters are not distinguished
  - Different from the 5 strings of two characters that are reserved
- How many different variable names are there in this version of BASIC?

- A string of 1 or 2 alphanumeric characters (number & letter)
- Begin with a letter
- Uppercase and lowercase letters are not distinguished
- Different from the 5 strings of two characters that are reserved

- Number of variables names containing 1 character (**V_1**)
  - $V_1 = 26$, because a one-character variable name must be a letter

- Number of variables names containing 2 characters (**V_2**)
  - For V_2, by the product rule there are 26 x 36 strings of length two that begin with a letter and end with an alphanumeric character
  - However, five of these are excluded, $V_2 = 26 \times 36 - 5 = 931$

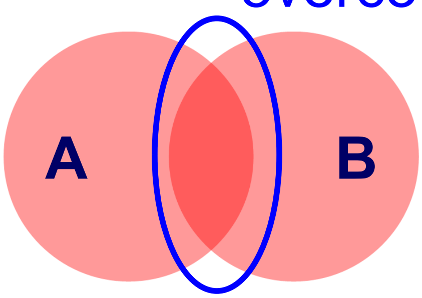
- Total number is $V_1 + V_2 = 26 + 931 = 957$

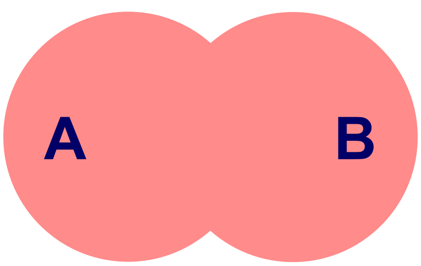
Basic Counting Principles:

Inclusion-Exclusion Principle

- Suppose that a task can be done in **A** or in **B ways**
- But some of the set of A ways to do the task are the **same** as some of the B ways to do the task

overcount





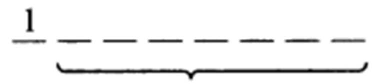
- **Avoid the overcount**

$$| A \cup B | = | A | + | B | - | A \cap B |$$

Basic Counting Principles: Inclusion-Exclusion Principle

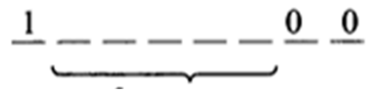
Example 1

- How many bit strings of length 8, either start with a 1 bit or end with the two bits 00?
- **Start with 1:** $2^7 = 128$ ways



- **End with 00:** $2^6 = 64$ ways

- Some of these strings are the same
  - The bit strings of length eight start with a 1 bit and end with the two bits 00



  - $2^5 = 32$
- **128 + 64 - 32 = 160**

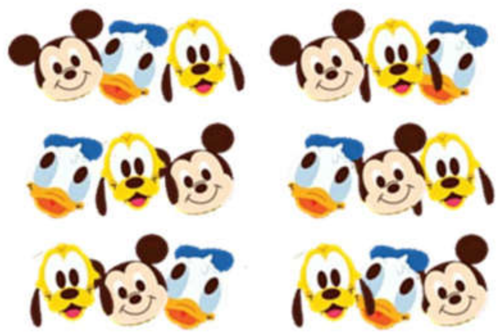
Basic Counting Principles: Inclusion-Exclusion Principle

Example 2

- A computer company receives **350** applications
- Suppose that
  - **220** majored in computer science
  - **147** majored in business
  - **51** majored both in computer science and in business
- How many of these applicants majored neither in computer science nor in business?
- Let A_1 : the set of students majored in computer science
  A_2 : the set of students majored in business
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316$
- **350 - 316 = 34** of the applicants majored neither in computer science nor in business

Permutation

- A **permutation** of a set of distinct n objects is an ordered arrangement of these objects



1^{st} 2^{nd} … r^{th} … n^{th}

n · (n-1) ·…· (n-r+1) ·…· 1 = **n!**

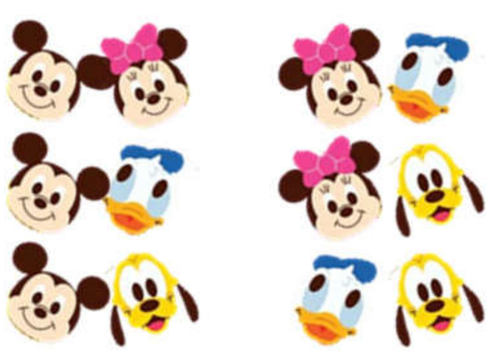
- **General Case**
  The ordering of r elements selected from n distinct elements is called **r-permutation**

$$_nP_r = P(n,r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

Combination

- The unordered selection of r elements from n distinct elements is called **r-combination**
  - It is a subset of the set with r elements

$$_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$



Combination

- $C(n, r) = C(n, n - r)$
- **Algebraic Proof**

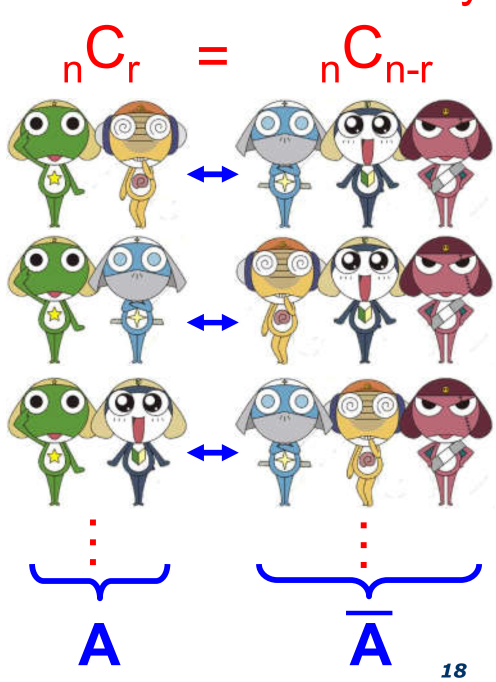
$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(n-(n-r))!(n-r)!}$$

$$= C(n, n-r)$$

Combination

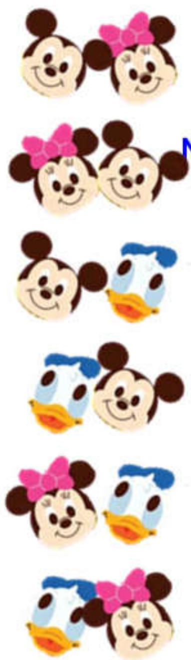
- **Combinatorial proof**
  - Using counting arguments to prove that both sides of the identity count the same objects but in different ways
  - Using combinatorial proof for $C(n, r) = C(n, n - r)$
    Suppose that **S** is a set with n elements.
    Every subset A of S with r elements corresponds to a subset of S with n - r elements, namely A Consequently,
    $C(n, r) = C(n, n - r)$



Basic Counting Principles

Permutation / Combination

- Proof



$$\underset{6}{{}_3P_2} = \underset{3}{{}_3C_2} \times \underset{2}{{}_2P_2}$$

Number of r-permutations of n elements

Number of r-permutations of r elements

$$P(n,r) = C(n,r) \cdot P(r,r)$$

Number of r-combinations of n elements

$$\frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Basic Counting Principles: Permutation / Combination

Example

- Your class has 10 students. How many different ways the committee can be set up:
  1. A committee of four ${}_{10}C_4$
  2. A committee of four and one person is to serve as chairperson ${}_{10}C_4 \cdot {}_4C_1$
  3. A committee of four and two co-chairpersons ${}_{10}C_4 \cdot {}_4C_2$
  4. Two committees: ${}_{10}C_4 \cdot {}_4C_2 \cdot {}_{10}C_3 \cdot {}_3C_1$
     - One with four members with two co-chairs
     - One with three members and a single chair

Pascal's Identity and Triangle

- **Pascal's Identity**
  Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

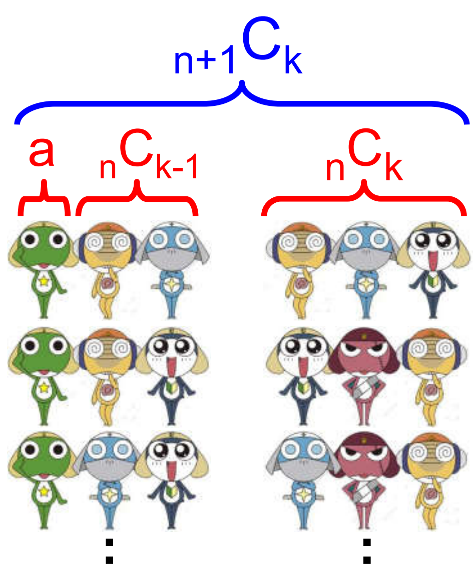
- **Pascal's triangle**
  A geometric arrangement of the binomial coefficients in a triangle
  - binomial coefficient is the sum of two adjacent binomial coefficients in the previous row

$_1C_0$ $\quad$ $_1C_1$ $\quad$ $_2C_1$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
⋮

Pascal's Identity and Triangle

- **Proof $_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$**
  - Suppose T is a set containing n + 1 elements
  - Let a be an element in T, and let S = T - {a}
  - There are $_{n+1}C_k$ subsets of T containing k elements
  - $_{n+1}C_k$ subsets contains either
    - ($_nC_{k-1}$) k - 1 elements of S and a, or
    - ($_nC_k$) k elements of S and not a
  - Therefore, $_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$



T = 

a =  S = 

Combinatorial Proof Example 2

Vandermonde's Identity

- Theorem: **Vandermonde's Identity**
  - Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

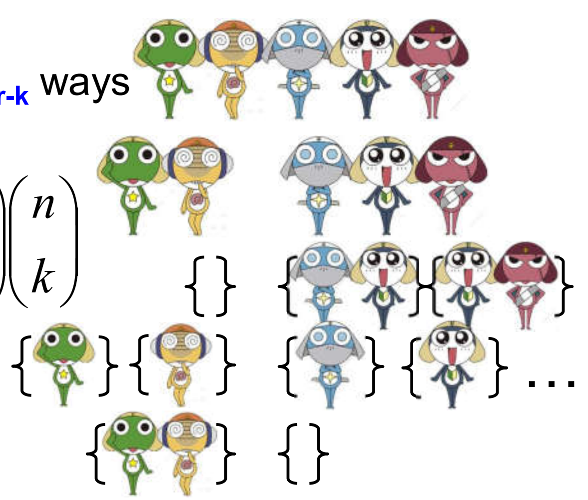
Combinatorial Proof Example 2

Vandermonde's Ide

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- **Proof**
  - Suppose: m items in a first set and n items in a second set
  - The total number of ways to pick r elements from the union of these sets is $_{m+n}C_r$
  - Another way is to pick k elements from the first set and then r - k elements from the second set, where k is an integer with 0 ≤ k ≤ r
    - There are $_mC_r \cdot {}_nC_{r-k}$ ways
  - Therefore,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$



$_{2+3}C_2$

$_mC_{r-k} \cdot {}_nC_k$

$_mC_0 \cdot {}_nC_2$

$_mC_1 \cdot {}_nC_1$

$_mC_2 \cdot {}_nC_0$

Combinatorial Proof Example 3
Theorem of Binomial Coefficients

- **Theorem**

  Let n and r be nonnegative integers with r ≤ n. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Combinatorial Proof Example 3
Theorem of Binomial Coefficients

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

${}_{4+1}C_{1+1}$	${}_{1}C_{1}$	${}_{2}C_{1}$	${}_{3}C_{1}$	${}_{4}C_{1}$
00011	11000	01100	00110	00011
00101		10100	01010	00101
00110			10010	01001
01001				10001
01010				
01100				
10001				
10010				
10100				
11000				

consider the possible locations of the final 1

Combinatorial Proof Example 3

Theorem of Binomial Coe

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- **Proof**:
  - Consider ${}_{n+1}C_{r+1}$ counts the bit strings of length n + 1 containing r + 1 ones

010100110...0 contain r+1 1s

n+1 bits

  - Another counting way is to consider the possible locations, named k, of the final 1
  - k should equal to r + 1, r + 2, . . ., or n + 1
    - r+1 ≤ k ≤ n+1

k

01110...10100

k-1 bits contain r 1s

Combinatorial Proof Example 3

Theorem of Binomial Coe

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

01110...10100 k

k-1 bits contain r 1s

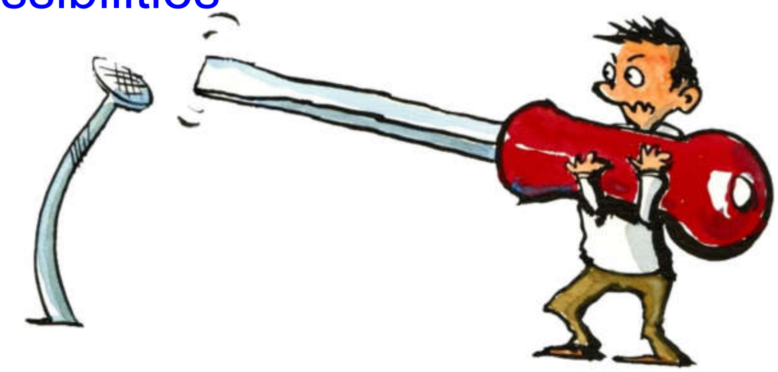
- Consider the first k-1 bits
  - In this k-1 bits, there should be r 1s
  - There are ${}_{k-1}C_r$ ways
  - Recall, r+1 ≤ k ≤ n+1

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

By the change of variables j = k - 1

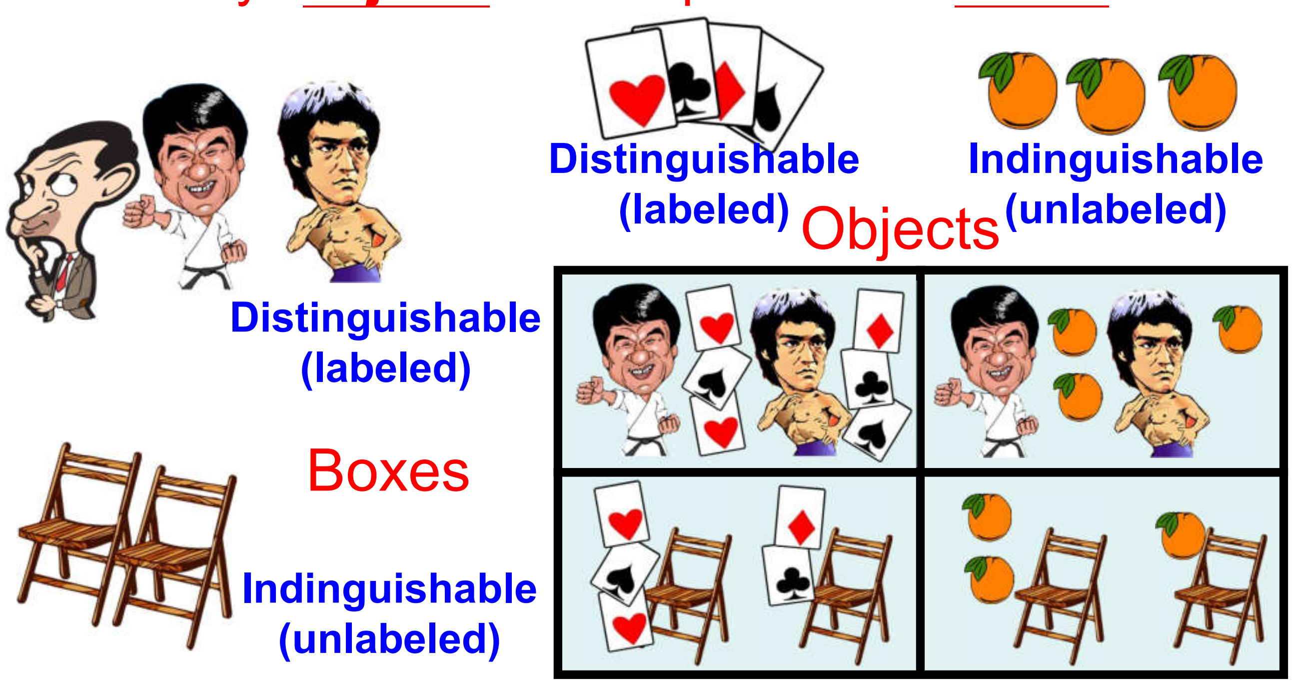
Counting Problems

- How to apply what you have learn to solve the counting problems?
  - Multiplication / Addition Principle
  - Inclusion-Exclusion Principle
  - Permutation / Combination
  - List all the possibilities



Counting Problems

- Many counting problems can be treated as the ways **objects** can be placed into **boxes**



Counting Problems

- General Algorithm
  - First check whether "Permutation / Combination" can be applied, otherwise, you need to "List all the possibilities"
  - Try to break down the problem into a subpart by using "Multiplication / Addition Principle" and "Inclusion-Exclusion Principle"

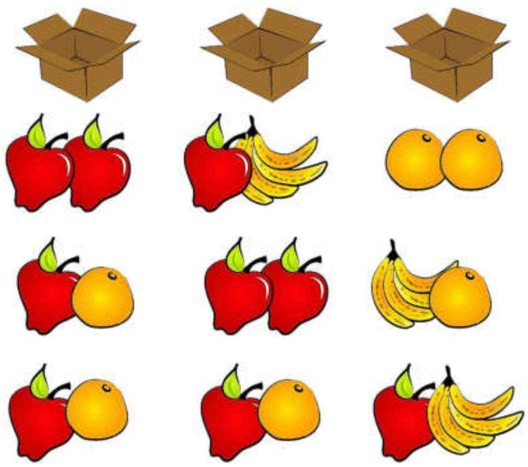
Counting Problems
Example 1

- There are five students (A, B, C, D & E) How many ways are there to arrange them:
  - into 5 seats? 5!
  - into 5 seats and A and B sit next to each other? 2 x 4! (AB and BA)
  - into 5 seats and A and B not sit next to each other? 5! - 2 x 4!
  - into a round table? 5! / 5 (each pattern counts 5 times)

Counting Problems

Example 2

- How many ways to put 3 apples, 2 oranges and 1 banana to 3 indistinguishable boxes and each box contains 2 items?



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Counting Problems

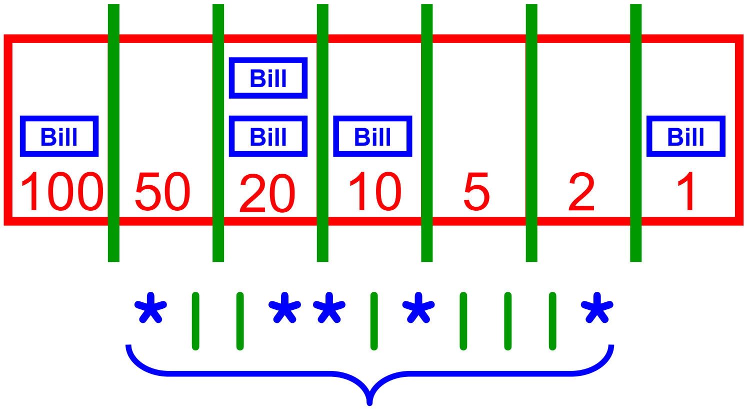
Example 3

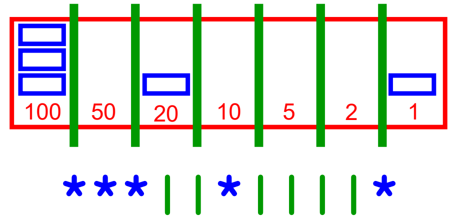
- How many ways are there to select 5 bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?

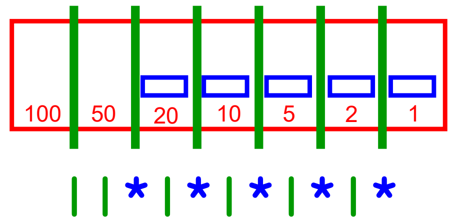
  Assume:
  - Order of selecting does not matter
  - Bills of each denomination are indistinguishable
  - At least five bills of each type

Example 3

Select five bills from \$1, \$2, \$5, \$10, \$20, \$50 and \$100







7 - 1 = 6 bars (lines between 7 boxes)

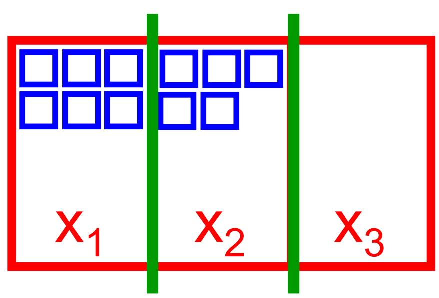
5 stars (5 bills)

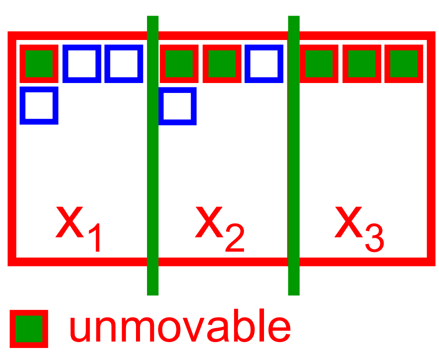
Total, 11 characters

$_{11}C_5 = 11! / (5!6!) = 462$

Example 4

- How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have?
  - where x_1, x_2, and x_3 are nonnegative integers.
    $n = 3, r = 11$
    $_{11+3-1}C_{11} = 78$
  - where x_1, x_2, and x_3 integers and $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$.
    $n = 3, r = 11 – 6 = 5$
    $_{5+3-1}C_5 = 21$





Example 5

- How many ways are there to pack 6 copies of the same book into 4 identical boxes, where a box can contain as many as six books?

- By listing all the possibilities

```
6, 0, 0 ,0      3, 3, 0, 0
5, 1, 0, 0      3, 2, 1, 0
4, 2, 0, 0      3, 1, 1, 1
4, 1, 1, 0      2, 2, 2, 0
                2, 2, 1, 1
```

- **There are 9 ways**

Generating Permutations & Combinations

- Sometimes permutations or combinations need to be generated but not just counted
  - E.g. all 3-combination for the set $\{a, b, \ldots, e\}$
  - {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, …

- How can we systemically generate all the combinations of the elements of a finite set?

Generating Combinations

- Recall that the bit string representation corresponding to a subset
  - For k^{th} position:
  - 1 : a_k is in the subset
  - 0 : a_k is not in the subset

b_3 b_2 b_1 b_0

1 0 1 1

Generating Combinations

Next Larger Bit String

- Algorithm: Generating the **next bit string** $(b_{n-1}, b_{n-2}, \ldots, b_1, b_0)$, where the current bit string is not equal to 11...11)
  1. $i = 0$
  2. while $b_i = 1$
     - 2.1 $b_i = 0$
     - 2.2 $i = i + 1$
  3. $b_i = 1$
- Treat it as adding “1” to a binary number

Example

- Find out the next combination using next larger bit string algorithm for

1 0 1 1

1. $i = 0$
2. while $b_i = 1$
   2.1 $b_i = 0$
   2.2 $i = i + 1$
3. $b_i = 1$

- Next:

1 1 0 0

b_3 b_2 b_1 b_0

Next Larger r-combinations

- Algorithm: Generating the next larger r-combinations after $\{a_1, a_2, \ldots, a_r\}$ by given a set $\{1, 2, 3, \ldots, n\}$

1. $i = r$
2. while $a_i = n - r + i$
   2.1 $i = i - 1$
3. $a_i = a_i + 1$
4. for $j = i + 1$ to r
   4.1 $a_j = a_i + j - i$

Steps 2–2.1: locate the last a_i ie $a_i \neq n - r + 1$

Step 3: add 1 to a_i

Steps 4–4.1: From a_{i+1} to a_r Assign new values

$\{a_1, a_2\}$
$\{1, 2, 3, 4\}$

{1, 2}
{1, 3}
{1, 4}
{2, 3}
{2, 4}
{3, 4}

Generating Combinations

Example 1

- Find the next larger 4-combination of the set {1, 2, 3, 4, 5, 6} after **{1, 2, 5, 6}**

1. $i = r$
2. while $a_i = n - r + i$
   2.1 $i = i - 1$
3. $a_i = a_i + 1$
4. for $j = i + 1$ to r
   4.1 $a_j = a_i + j - i$

- $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_4 = 6$
- The last a_i such that $a_i \neq n - r + 1$ is a_2 **(i = 2)**

$a_4 = 6 = 6 - 4 + 4$

$a_3 = 5 = 6 - 4 + 3$

$a_2 = 2 \neq 6 - 4 + 2$

- Next larger 4-combination
  - $\boldsymbol{a_2} = a_2 + 1 = 2 + 1 = 3$
  - $\boldsymbol{a_3} = a_2 + j - i = 3 + 3 - 2 = 4$
  - $\boldsymbol{a_4} = a_2 + j - i = 3 + 4 - 2 = 5$
- Hence : **{1, 3, 4, 5}**

Generating Combinations

Example 2

- List all 3-combination for the set $\{a, b, \ldots, e\}$
  - Assume $\{a, b, \ldots, e\} = \{1, 2, \ldots, 5\}$
  - For all $\{a_1, a_2, a_3\}$

1. $\{a, b, c\}$
2. $\{a, b, d\}$
3. $\{a, b, e\}$
4. $\{a, c, d\}$
5. $\{a, c, e\}$
6. $\{a, d, e\}$
7. $\{b, c, d\}$
8. $\{b, c, e\}$
9. $\{b, d, e\}$
10. $\{c, d, e\}$

1. $i = r$
2. while $a_i = n - r + i$
   2.1 $i = i - 1$
3. $a_i = a_i + 1$
4. for $j = i + 1$ to r
   4.1 $a_j = a_i + j - i$

Generating Permutations

- Any set can be placed in one-to-one correspondence with the set {1, 2, 3, ..., n}
  - The permutations of any set of n elements can be listed by generating the permutations of the n smallest positive integers
- The algorithms based on the lexicographic (or dictionary) ordering is discussed

Generating Permutations

- Algorithm: Generating the **next permutation** of $(a_1, a_2, ..., a_n)$ in Lexicographic Order by given permutation is $\{1, 2, ..., n\}$, where $(a_1, a_2, ..., a_n)$ is not equal to $(n, n-1, ..., 2, 1)$

1. $j = n - 1$
2. while $a_j > a_{j+1}$
   2.1 $j := j - 1$
3. $k = n$
4. while $a_j > a_k$
   4.1 $k = k - 1$
5. interchange a_j and a_k
6. $r = n$
7. $s = j + 1$
8. while $r > s$
   8.1 interchange a_r and a_s
   8.2 $r = r - 1$ and $s = s + 1$

Steps 2–2.1: j is the largest subscript with $a_j < a_{j+1}$

Steps 4–4.1: k is the largest subscript with $a_j < a_k$

Steps 6–8.2: Sort the number after the j^{th} position in ascending order

$\{a_1, a_2, a_3\}$
{1, 2, 3}
{1, 2, 3}
{1, 3, 2}
{2, 1, 3}
{2, 3, 1}
{3, 1, 2}
{3, 2, 1}

Example

- What is the next permutation in lexicographic order after **362541**?

- The last pair of a_j and a_{j+1} where $a_j < a_{j+1}$ is a_3 **= 2** and a_4 **= 5**
- The least integer to the right of 2 that is greater than 2 is a_s **= 4**
- Exchange a_j and a_s
  - Hence, 4 is placed in the third position
- 5, 2, 1 are placed in order in the last three positions
- Hence, the next permutation is **364125**

r-Permutations

- How can we list all r-permutations from a set {1, 2, 3, . . . , n}?

1. Use “next larger r-combinations” lists all r-combinations
2. For each r-combination, use n-permutation to list all permutations

r-combination

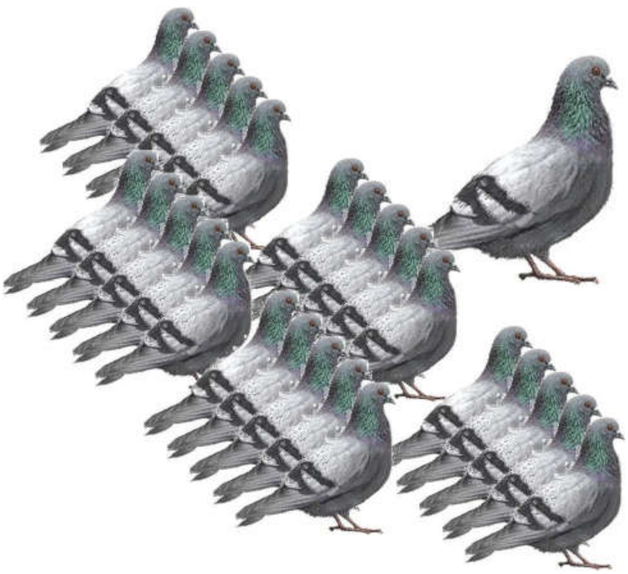
$\{a_1, a_2, a_3\}$

{1, 2, 3, 4}

r-combination	n-permutation
{1, 2, 3}	**{1, 2, 3}**
{1, 2, 4}	**{1, 3, 2}**
{1, 3, 4}	**{2, 1, 3}**
{2, 3, 4}	**{2, 3, 1}**
	{3, 1, 2}
	{3, 2, 1}

Pigeonhole Principle

- Suppose that a flock of 26 pigeons flies into a set of 25 pigeonholes to roost
- What can we conclude?



Pigeonhole Principle

- A least one of these 25 pigeonholes must have at least two pigeons in it
  - Because there are 26 pigeons but only 25 pigeonholes
- This is **Pigeonhole Principle**

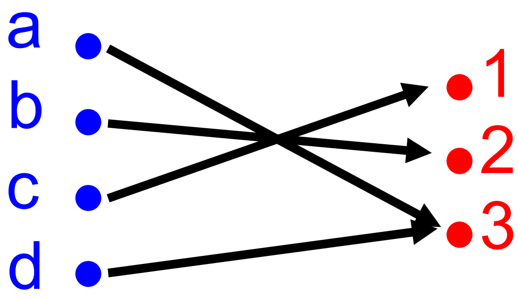
Pigeonhole Principle

- **Pigeonhole Principle**
  If k is a positive integer and **k + 1** or more objects are placed into **k** boxes, then there is at least one box containing two or more of the objects
  - Also called **the Dirichlet Drawer Principle**
    the nineteenth-century German mathematician Dirichlet

- **Proof by contraposition ($p \to q \equiv \neg q \to \neg p$)**
  - Suppose that none of the k boxes contains more than one object
  - Then the total number of objects would be at most k
  - This is a contradiction

Pigeonhole Principle

- **Corollary**
  A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one



Pigeonhole Principle

- **Example 1**
  How many words we should have if there must be at least two that begin with the same letter?
  - 27 English words, because 26 letters in the English alphabet

- **Example 2**
  How many people we should have if there must be at least two with the same birthday?
  - 367 people because 366 possible birthdays

Generalized Pigeonhole Principle

- **Pigeonhole Principle** states that if k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- How about if we have
  - 2k + 1 objects?
  - 3k + 2 object?
  - nk + 1 object?

Generalized Pigeonhole Principle

- <u>Generalized Pigeonhole Principle</u>
  If N objects are placed into k boxes, then there is at least one box containing at least ⌈N/k⌉ objects

- **Proof by Contradiction**
  - Suppose that none of the boxes contains more than ⌈N/k⌉ - 1 objects
  - The total number of objects is at most

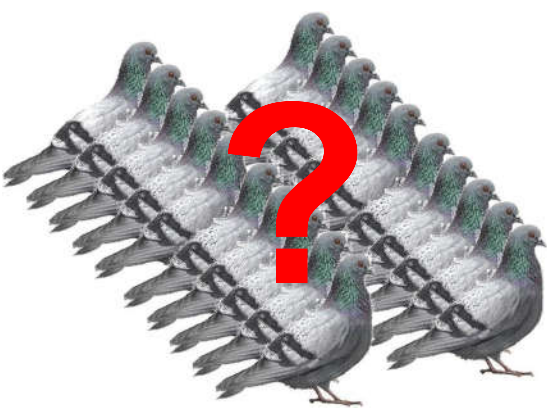
$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

⌈N/k⌉ < (N/k) + 1

  - This is a contradiction because there are a total of N objects

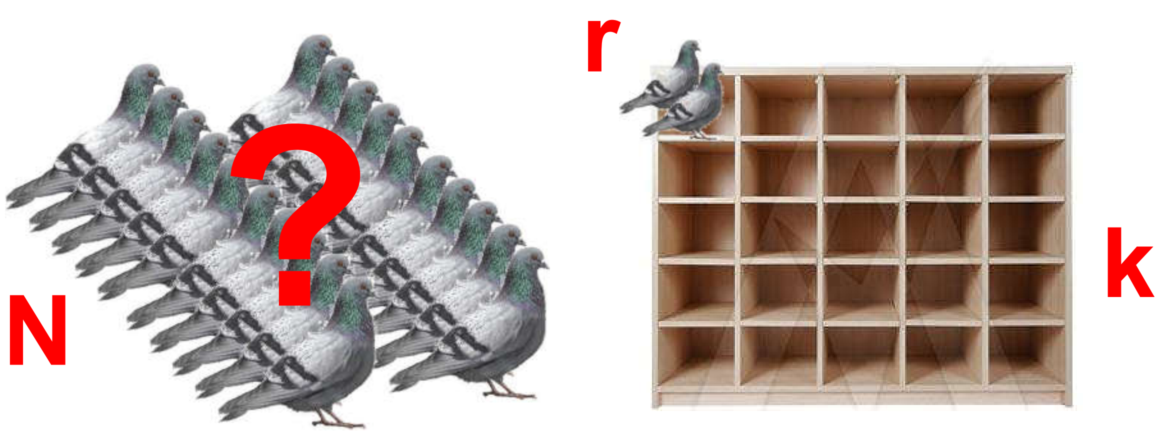
Generalized Pigeonhole Principle

- A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes



Generalized Pigeonhole Principle

- According to generalized pigeonhole principle, when we have N objects, there must be at least r objects in one of the k boxes as long as $\lceil N/k \rceil \geq r$
  - N, where $N = k(r - 1) + 1$, is the **smallest integer** satisfying $\lceil N/k \rceil \geq r$



Generalized Pigeonhole Principle

- $\lceil N/k \rceil \geq r$, $N = k(r - 1) + 1$, is the **smallest integer** satisfying $\lceil N/k \rceil \geq r$
- Could a smaller value of N suffice?
- **No**
  - If $k(r - 1)$ objects
  - We could put $r - 1$ of them in each of the k boxes
  - No box would have at least r objects

Example 1

$\lceil N/k \rceil \geq r$
$N = k(r - 1) + 1$

- How many people out of 100 people were born in the same month?
- N = 100
- k = 12
- r = ?
- $\lceil 100/12 \rceil = 9$ who were born in the same month

Generalized Pigeonhole Principle

Example 2

$\lceil N/k \rceil \geq r$
$N = k(r - 1) + 1$

- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?
- Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.
- Different phone numbers for NXX-XXXX is $8 \times 10^6 = 8{,}000{,}000$
- N = 25,000,000, k = 8,000,000
- At least $\lceil 25{,}000{,}000 / 8{,}000{,}000 \rceil = 4$ of them must have identical phone numbers
- Hence, at least four area codes are required

Example 3

$\lceil N/k \rceil \geq r$

$N = k(r - 1) + 1$

- Show that among any n + 1 positive integers not exceeding $2n$ there must be an integer that divides one of the other integers
- Assume we have n + 1 integers $a_1, a_2, \ldots, a_{n+1}$
- Let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n + 1$,

  where k_j is a nonnegative integer and $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$
- According to pigeonhole principle, because only n odd positive integers less than $2n$, two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal
- Let q be the common value of q_i and q_j, then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$
- It follows that if $k_i < k_j$, then a_i divides a_j; otherwise a_j divides a_i

$$\frac{a_j}{a_i} = \frac{2^{k_j} q}{2^{k_i} q} = 2^{k_j - k_i}$$

Applications: Subsequence

- Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers.
- A **subsequence** of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ where $1 < i_1 < i_2 < \ldots < i_m < N$

Example

- Example:
  - $a_1, a_2, ..., a_5$ = 5, 8, 2, 3, 1
  - 5, 3, 1 is a subsequence? a_1, a_4, a_5 ✓
  - 8, 1 is a subsequence? a_2, a_5 ✓
  - 2, 3, 5, 8 is a subsequence? a_3, a_4, a_1, a_2 ✗

Applications: Subsequence

- A sequence is called **strictly increasing** if each term is larger than the one that precedes it
- A sequence is called **strictly decreasing** if each term is smaller than the one that precedes it

Applications: Subsequence

- **Theorem**
  Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing

- Example
  - Given a sequence: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7
    - 10 term = $3^2 + 1$
  - What is the length of the longest in / decreasing subsequences? **n+1 = 4**
    - Increasing sequence
      - 1, 4, 6, 12
      - 1, 4, 6, 7
      - 1, 4, 6, 10
      - 1, 4, 5, 7
    - Decreasing sequence
      - 11, 9, 6 ,5

Proof

- Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers

- Associate an ordered pair (i_k, d_k) to the term a_k, where
  - i_k is the length of the longest increasing subsequence starting at a_k
  - d_k is the length of the longest decreasing subsequence starting at a_k

5, 8, 2, 3, 1

$(i_1, d_1) = (2, 3)$ $(i_4, d_4) = (1, 2)$

Applications: Subsequence
Proof

5, 8, 2, 3, 1
(2, 3) (1, 2)

- Suppose no increasing or decreasing subsequences is longer than n
- i_k and d_k are both positive integers less than or equal to n, for k = 1, 2, . . . , $n^2 + 1$
- By the product rule, n^2 possible ordered pairs for (i_k, d_k)
- By the pigeonhole principle two of $n^2 + 1$ ordered pairs are equal
- Therefore, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$

Applications: Subsequence
Proof

…, a_s, …, a_t, …

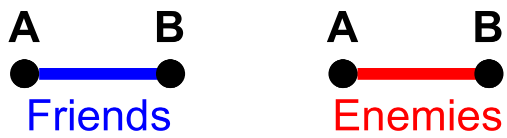
There exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$

5, 8, 2, 3, 1
(2, 3) (1, 2)

- We will show that this is impossible
- Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$
- If $a_s < a_t$, then, because $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking as followed by an increasing subsequence of length it beginning at a_t
- **This is a contradiction**
- Similarly, if $a_s > a_t$, it can be shown that d_s must be greater than d_t, which is a **contradiction**

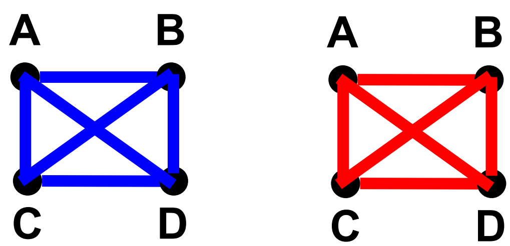
Applications: Ramsey Theory

- **Ramsey theory**, after the English mathematician F. Ramsey, deals with the distribution of subsets of elements of sets
  - **Two people either friends or enemies**



  - **Mutual Friend/Enemies**

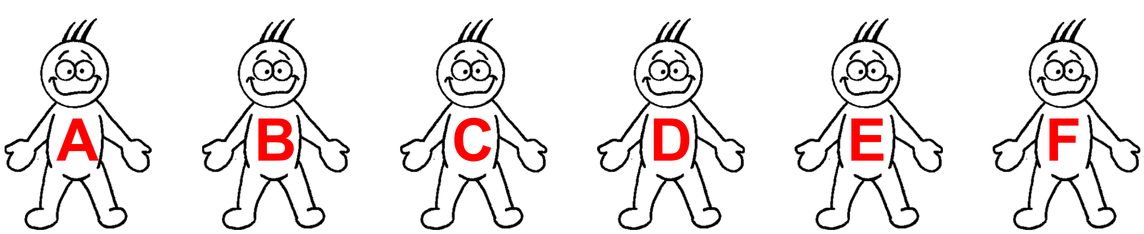
A B C D are mutual friends/enemies



Applications: Ramsey Theory

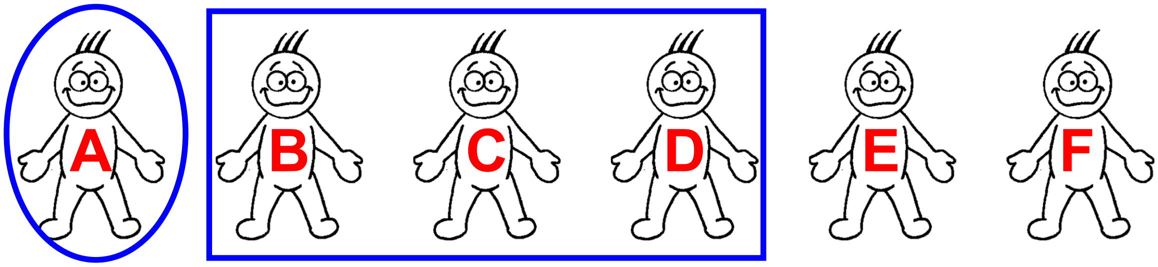
Example 1

- Assume that in a group of six people
- Show that there are either three mutual friends or three mutual enemies in the group



Example 1

- Let **A** be one of the six people
- According to pigeonhole principle ($\lceil 5/2 \rceil = 3$),
  A at least has three friends, or three enemies
- **Former Case**: suppose that B, C, and D are friends
  - If any two of these three people are friends, then these two and A form a group of three mutual friends
  - Otherwise, B, C, and D form a set of three mutual enemies
- **Similar to the latter case**



Applications: Ramsey Theory

- **Ramsey number R(m, n)**
  - The minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies
  - **m** and **n** are positive integers greater than or equal to 2
- Example
  - What is **R(3, 3)**?
    - Answer should be **6**
    - In a group of five people where every two people are friends or enemies, there may not be three mutual friends or three mutual enemies

Applications: Ramsey Theory

- 5 people cannot guarantee having 3 mutual friends/enemies