Discrete Mathematic

Chapter 5: Relation

5.6
Partial Orderings

Dr Patrick Chan
School of Computer Science and Engineering
South China University of Technology

Agenda

- Partial Order
- Total Order
- Lexicographic Order
- Hasse Diagrams
- Minimal/Maximal Element
- Least/Greatest Element
- Lower/Upper Bound
- Greatest Lower/Least Upper Bound

What is Order?

What is Order?

- **Equivalence (=)** concept is discussed
- The abstraction of the following relations will be discussed in this chapter
  - Bigger or Equal / Smaller or Equal (≤, ≥)
  - Bigger / Smaller (<, >)

What is Order?

- What properties "≤" or "≥" should have?

  Reflexive ~~Irreflexive~~ Transitive

  ~~Symmetric~~ ~~Asymmetric~~ Antisymmetric

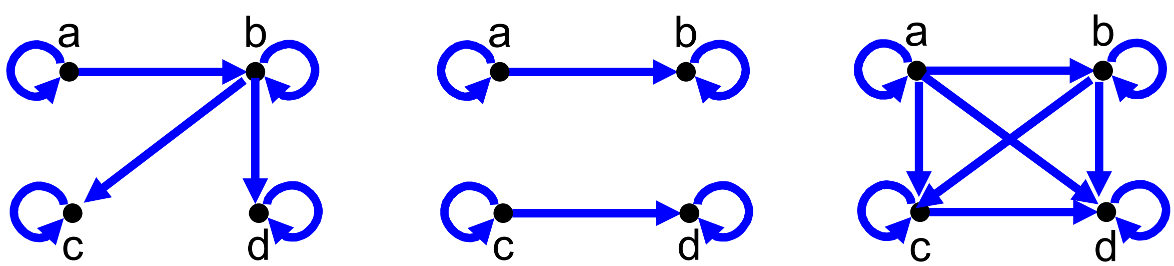
- What properties "<" or ">" should have?

  ~~Reflexive~~ Irreflexive Transitive

  ~~Symmetric~~ Asymmetric Antisymmetric

Partially Ordered Set

- Definition
  Let R be a relation on A.
  Then R is a partial order iff R is
  - **Reflexive**
    $\forall a\ (\ (a, a) \in R\ )$
  - **Antisymmetric**
    $\forall a\ \forall b\ (\ ((a, b)\in R \wedge (b, a)\in R) \rightarrow (a = b)\ )$
  - **Transitive**
    $\forall a \forall b \forall c\ (\ ((a,b)\in R \wedge (b,c)\in R) \rightarrow ((a,c)\in R))$



Not Partial Order **Partial Order** **Partial Order**

Partially Ordered Set

- When R is a partial order in A, **(A, R)** is called a partially ordered set or a **poset**
- Recall, aRb denotes that $(a,b) \in R$
- If R is a partial ordering relation

  $a \preccurlyeq b$ denotes that $(a,b) \in \preccurlyeq$
- $(A, \preccurlyeq)$ is a poset

Partially Ordered Set
Example 1

- Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers
- **Reflexive**

  $a \geq a$ for every integer a
- **Antisymmetric**

  If $a \geq b$ and $b \geq a$, then $a = b$
- **Transitive**

  $a \geq b$ and $b \geq c$ imply that $a \geq c$
- $(Z, \geq)$ is a poset

Partially Ordered Set

Example 2

- **(Z^+, |)** is a poset

  The divisibility relation | is a partial ordering on the set of positive integers.

  - i.e. a divides b

- **(P(S), $\subseteq$)** is a poset

  The inclusion relation $\subseteq$ is a partial ordering on the power set of a set S

  - i.e. a is the subset of b

Comparability

- The elements a and b of a poset(S, $\preccurlyeq$) are called comparable if either a $\preccurlyeq$ b or b $\preccurlyeq$ a
- Otherwise, a and b are imcomparable
- Example
  - In the **poset (Z^+, |)**,
    - Are 3 and 9 comparable? **Yes, since 3 | 9**
    - Are 5 and 7 comparable? **No**

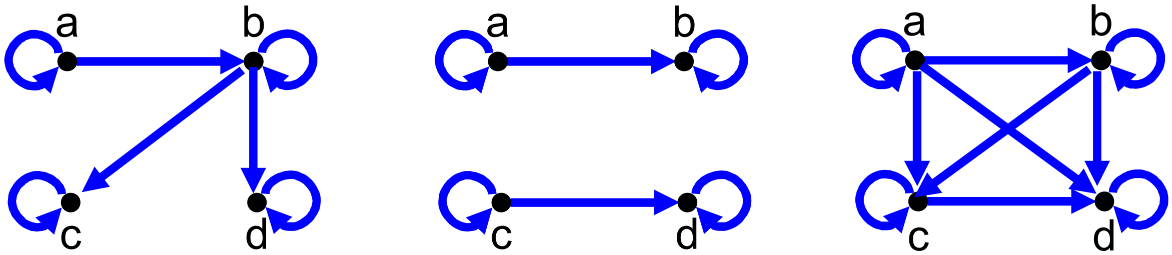
Non-Strict & Strict Partial Order

- Non-strict (or reflexive) Partial Order $\preccurlyeq$
  - Property: **Reflexive**, Antisymmetric, Transitive
- Strict (or irreflexive) Partial Order $\prec$
  - i.e. $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$
  - Property: **Irreflexive**, Antisymmetric, Transitive

    (Irreflexive + Antisymmetric: **asymmetric**)
- Generally, partial order refers to $\preccurlyeq$

Total Ordered

- If (S, R) is a poset and every two elements are comparable, S is called a total ordered or linear ordered or simple ordered set
- In this case (S, R) is called a chain



Not Partial Order	Partial Order	Partial Order
Not Total Order	Not Total Order	Total Order

Example

- **Poset $(Z, \leq)$?** Totally Ordered
  - Since $a \leq b$ or $b \leq a$ whenever a and b are integers
- **Poset $(Z^+, |)$?** Not totally Ordered
  - It contains elements that are incomparable, such as 5 and 7
- **Poset $(P(S), \subseteq)$**, where S is a set Not totally Ordered
  - It may not be the case that $A \subseteq B$ or $B \subseteq A$

Lexicographic Order

- What is the order of a letter? $A \preccurlyeq C$? $C \preccurlyeq A$?
  - Alphabetical order
- What is the order of a word?
  - **Lexicographic Order**
  - Generalization of Alphabetical order

discrete $\preccurlyeq$ discreet ?

discreetness $\preccurlyeq$ discreet ?

Lexicographic Order
Special Case

- **Lexicographic Order** is a generalization of the way the alphabetical order of words is based on the alphabetical order of letters
  - Also known as lexical order, dictionary order, alphabetical order or lexicographic(al) product
- Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$ we construct an Lexicographic Order $\preccurlyeq$ on $A_1 \times A_2$:

  $<x_1,y_1> \preccurlyeq <x_2,y_2>$ iff $x_1 \prec_1 x_2$ or $(x_1 = x_2$ and $y_1 \preccurlyeq_2 y_2)$

Lexicographic Order
Special Case: Example

For $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$
$<x_1,y_1>$ R $<x_2,y_2>$ iff $x_1 \prec_1 x_2$ or $(x_1 = x_2$ and $y_1 \preccurlyeq_2 y_2)$

- Let $A_1 = A_2 = Z^+$ and $R_1 = R_2$ = 'divides'.
- If the following relation is Lexicographic Order R?
  - (2, 4) R (2, 8)? Condition 1 ✗ 2 = 2; Condition 2 ✓ 4 divides 8
  - ~~(2, 4) R (2, 6)?~~ Condition 1 ✗ 2 = 2; Condition 2 ✗ 4 does not divide 6
  - (2, 4) R (4, 5)? Condition 1 ✓ 2 divides 4; Condition 2 ✗ 4 does not divide 5

1. $(x_1 \neq x_2)$ and x_1 divides x_2?
2. $(x_1 = x_2)$ and $(y_1 \neq y_2)$ and (y_1 divides y_2)?

Lexicographic Order

General Case

- Given $(A_1, \preccurlyeq_1), (A_2, \preccurlyeq_2), \ldots, (A_n, \preccurlyeq_n)$
- The Lexicographic Order $\prec$ on multiple Cartesian products: $A_1 \times A_2 \times A_3 \times \ldots \times A_n$

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

- iff
  - If $a_1 \prec_1 b_1$, or
  - if there is an integer i>0 such that $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} \prec_{i+1} b_{i+1}$

For two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$
$<x_1,y_1> \prec <x_2,y_2>$ iff $x_1 \prec_1 x_2$ or ($x_1 = x_2$ and $y_1 \prec_2 y_2$)

Lexicographic Order

General Case: Example

- Let $A_1 = A_2 = \ldots = A_n = Z^+$ and $R_1 = R_2 = \ldots = R_i =$ 'divides'
- If the following relation is Lexicographic Order R?

- (2, 3, 4, 5) R (2, 3, 8, 2)?

i=1 $2 \prec_1 2$?

i=2 $2 = 2$ and $3 \prec_2 3$?

i=3 $2 = 2$ and $3 = 3$ and $4 \prec_3 8$? ✔

- (2, 3, 4, 5) R (3, 6, 8, 10)?

$2 \prec_1 3$?

$2 = 3$ and $3 \prec_2 6$? ✘

Do not need to check the rest as $a_1 \neq b_1$

- If $a_1 \prec_1 b_1$, or
- if there is an integer i>0 such that $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} \prec_{i+1} b_{i+1}$

Lexicographic Order
String

- Lexicographic order is applied to strings of symbols where there is an underlying 'alphabetical' order

- Consider the different strings $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_n$ on a partial ordered set S
- Let $t = \min(m, n)$, the definition of lexicographic ordering for string is that the string $a_1a_2\ldots a_m$ is less than $b_1b_2\ldots b_n$ if and only if
  - ($(a_1,a_2,\ldots,a_t) = (b_1,b_2,\ldots,b_t)$ and $m < n$) or
    $(a_1,a_2,\ldots,a_t) \le (b_1,b_2,\ldots,b_t)$

lexicographic ordering using 'alphabetical' order

- If $a_1 \prec_1 b_1$, or
- if there is an integer i>0 such that $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} \prec_{i+1} b_{i+1}$

- $(a_1,a_2,\ldots,a_t) = (b_1,b_2,\ldots,b_t)$ and m<n
  or
- $(a_1,a_2,\ldots,a_t) \le (b_1,b_2,\ldots,b_t)$

- Consider the set of strings of lowercase English letters. $t = \min(m, n)$
  - "discrete" length = 8 **t = 8** alphabetical order: e < t
    "discreet" length = 8 discreet $\prec$ discrete
  - "discreet" length = 8 **t = 8**
    "discreetness" length = 12 discreet $\prec$ discreetness
  - "fiscrete" length = 8 **t = 8** alphabetical order: d < f
    "discreteen" length = 12 discreteen $\prec$ fiscrete

Lexicographic Order
String: Example

- $(a_1,a_2,\ldots,a_t) = (b_1,b_2,\ldots,b_t)$ and m<n
  or
- $(a_1,a_2,\ldots,a_t) \leq (b_1,b_2,\ldots,b_t)$

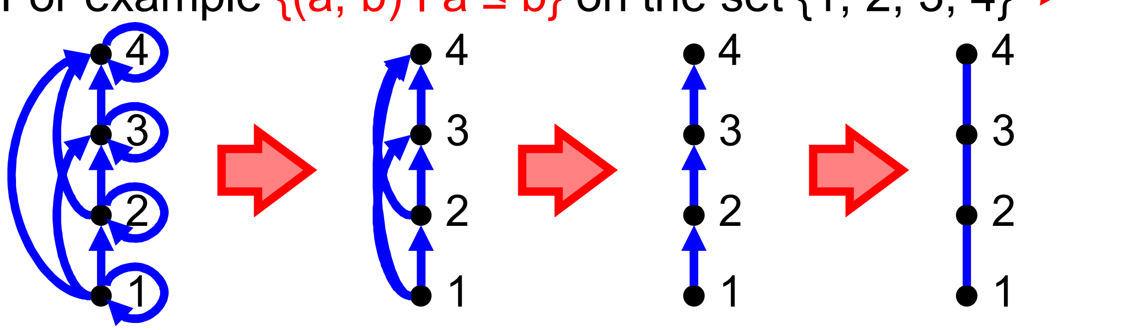
- Find the lexicographic ordering of the bit strings 0,01,11,001,010,011,0001 and 0101 based on the ordering $0 \prec 1$

0		0
01	0	0≤01
11	0≤01	0≤01≤11
001	0≤01≤11	0≤001≤01≤11
010	0≤001≤01≤11	0≤001≤01≤010≤11
0001	0≤001≤01≤010≤11	0≤0001≤001≤01≤010≤11

Hasse Diagrams

It is Hasse Diagram

- Show the partial ordering using a graph
  - For example {(a, b) I a ≤ b} on the set {1, 2, 3, 4}



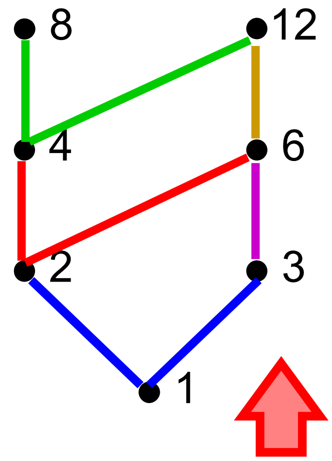
- The graph is too complicated and try to simplify it:
  - A partial ordering must be reflexive:
    the loops are not necessary
  - A partial ordering must be transitive:
    some edges can be removed
  - By assuming all edges are pointed upward,
    the direction of edges is not necessary

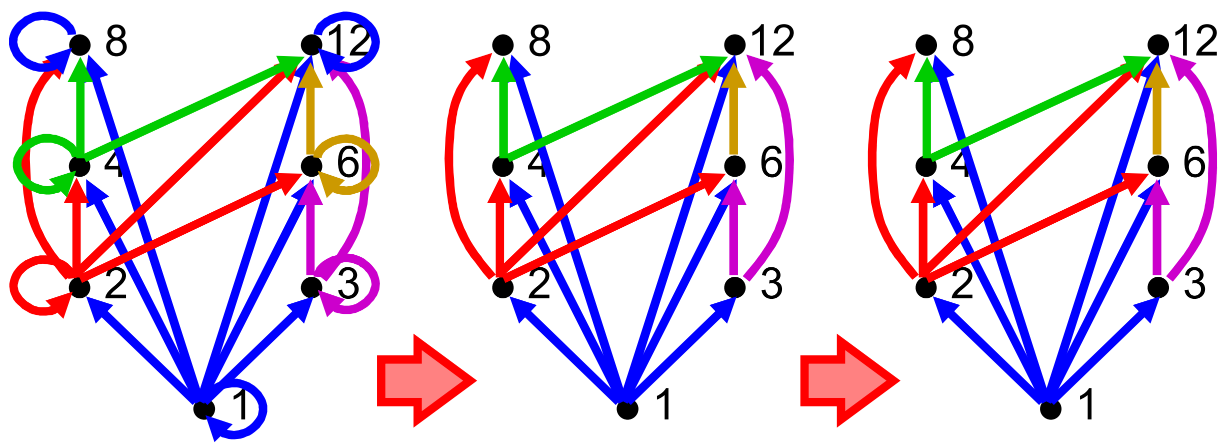
Hasse Diagrams

- To construct a Hasse diagram:
  1. Construct a digraph representation of the poset (A, R) so that all arcs point up (except the loops).
  2. Eliminate all loops
  3. Eliminate all redundant arcs
     - Start to eliminate from the top
  4. Eliminate the direction of the edge

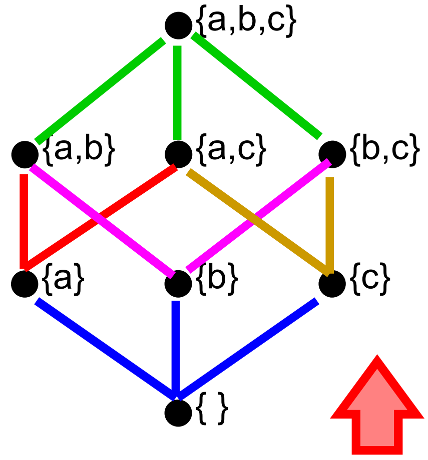
Hasse Diagrams
Example 1

- Draw the Hasse diagram representing the partial ordering { (a,b) | a divides b } on A={1,2,3,4,6,8,12}

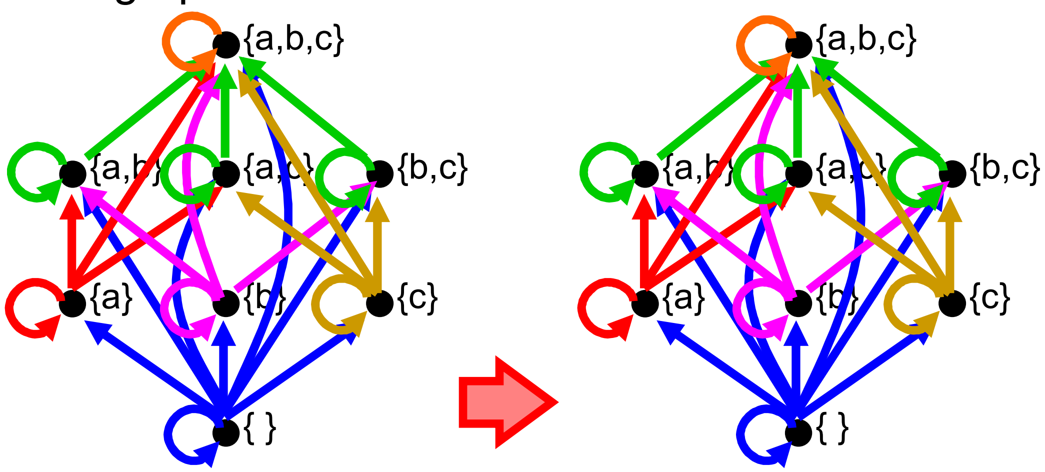




☺ Small Exercise ☺

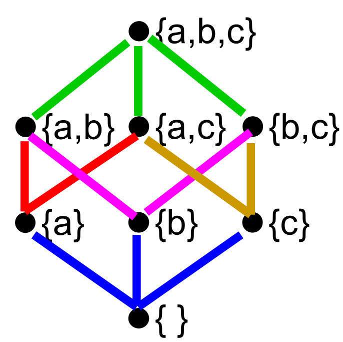


- Construct the Hasse diagram of (P({a, b, c}), ⊆)
- The elements of P({a, b, c}) are ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}
- The digraph is



Covering Relation

- Let (S, ≼) be a poset. (x,y) such that y cover x is called the covering relation of (S, ≼) if x ≺ y and there is no element z ∈ S such that x ≺ z ≺ y

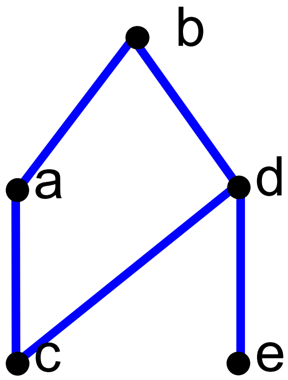


- Example
  - For (P({a, b, c}), ⊆), if it is a covering relation?
    - ({a,b}, {a})? ✗ {a} ≺ {a,b}
    - ({a}, {a,b})? ✓
    - ({ }, {a,b})? ✗ { } ≺ {a} ≺ {a,b} or { } ≺ {b} ≺ {a,b}
    - ({a}, {a})? ✗ {a} = {a}

Minimal & Maximal Elements

- Let (A, R) be a poset and $S \subseteq A$.
  s (b) in S is a minimal element (maximal element) of S iff there does not exist an element x in S such that xRs (bRx)

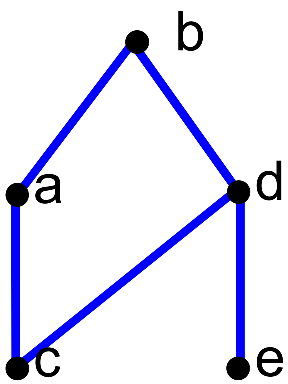
	maximal element	minimal element
{ a, b, c}	b	c
{ a, b, c, d, e}	b	c, e



Least & Greatest Elements

- Let (A, R) be a poset and $S \subseteq A$.
  s (b) in S is a least element (greatest element) of S iff sRx (xRb) for every x in S
- It is unique if it exits

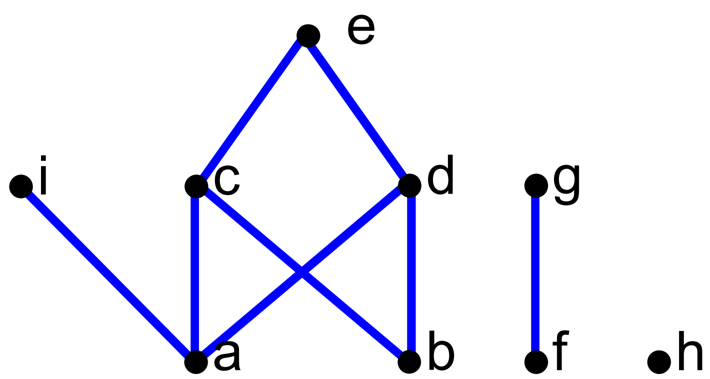
	Greatest element	Least element
{ a, b, c}	b	c
{ a, b, c, d, e}	b	/



☺ Small Exercise ☺

- **Minimal element: a b f h**
  - Not exist an element x in S such that xRs
- **Maximal element: e i g h**
  - Not exist an element x in S such that bRx
- **Least element: /**
  - sRx for every x in S
- **Greatest element: /**
  - xRb for every x in S

S = {a,b,c,d,e,f,g,h,i}



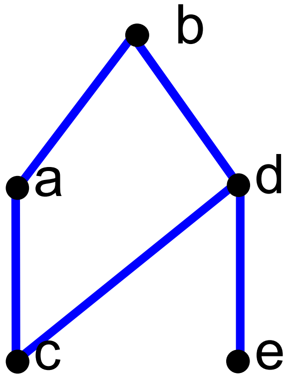
Well Ordered

- A chain (A, R) is well-ordered iff every nonempty subset of A has a least element
- Examples:
  - (Z, ≤) is a chain but not well-ordered
    - Z does not have least element
  - (N, ≤) is well-ordered
  - (N, ≥) is not well-ordered

Upper & Lower Bound

- Let (A, R) be a poset and $S \subseteq A$.
  s (b) in A is an lower bound (upper bound) of S iff sRx (xRb) for every x in S

	Upper Bound	Lower Bound
{a, d}	b	c
{c, d, e}	d, b	/

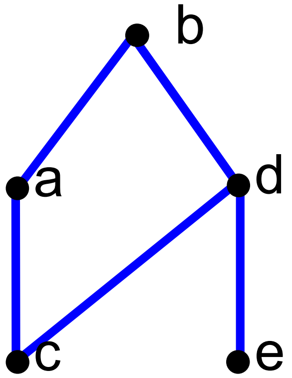


Greatest Lower & Least Upper Bounds

- Let (A, R) be a poset and $S \subseteq A$.
  s (b) is the least upper bound (greatest lower bound), denoted lub(S) (glb(S)), iff s (b) is an upper bound (lower bound) for S and sRx (yRb) for all other upper bounds x (lower bounds y) of S
- It is unique if it exits

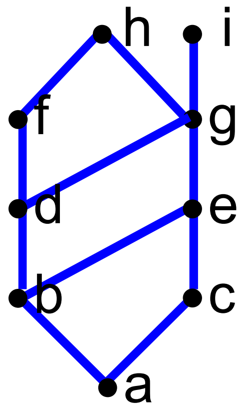
	Upper Bound	Lower Bound
{a, d}	b	c
{c, d, e}	d, b	/

	Least Upper Bound	Greatest Lower Bound
{a, d}	b	c
{c, d, e}	d	/



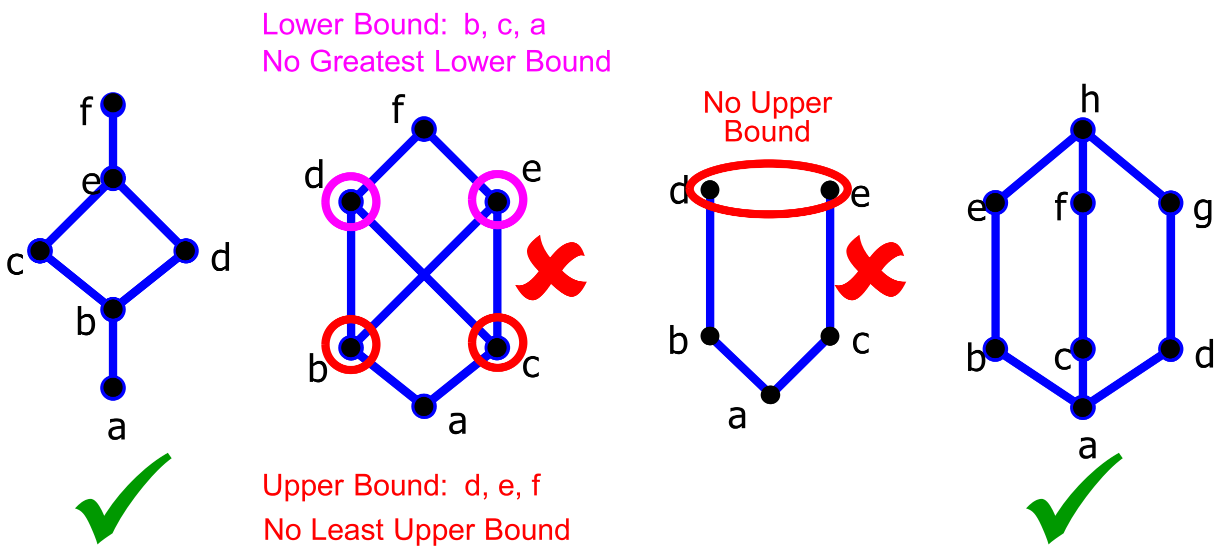
☺ Small Exercise ☺

	{a,b,c}	{b,c,d,g}	{h,i}
Minimal Element	a	b,c	h,i
Maximal Element	b,c	g	h,i
Least Element	a	/	/
Greatest Element	/	g	/
Lower Bound	a	a	a,b,c,e,d,g
Upper Bound	e,g,i,h	g,i,h	/
Greatest Lower Bound	a	a	g
Least Upper Bound	e	g	/



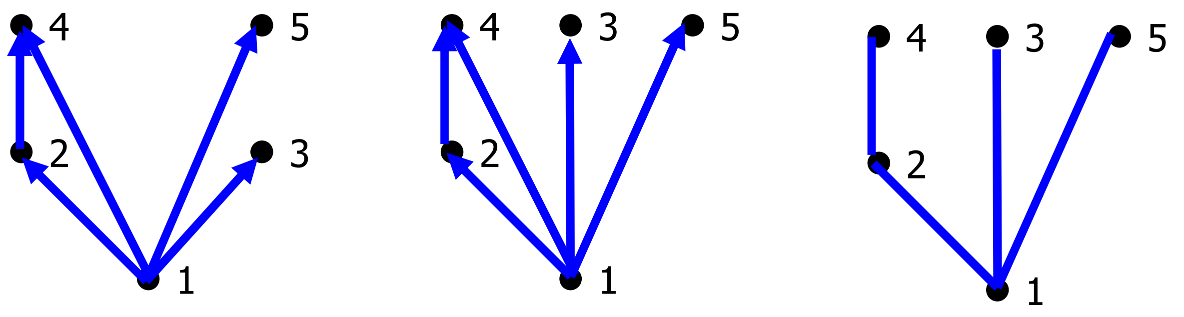
Lattice

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**



Lattice: Example

- **Poset ({1,2,3,4,5}, |) ?** **Not Lattice**
  - 2,3 have no upper bounds in {1,2,3,4,5}
- **Poset ({1,2,4,8,16}, |) ?** **Lattice**



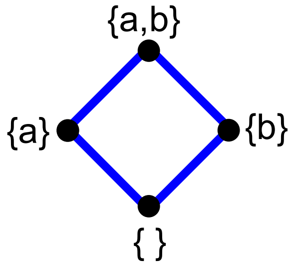
Lattice: Theorem

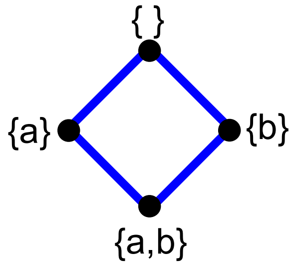
- **Theorem:** If L is a lattice, least upper bound and greatest lower bound of a and b can be defined as **a** $\vee$ **b** and **a** $\wedge$ **b**, respectively. $\vee$ and $\wedge$ satisfy the following properties for a,b,c $\in$ L.
  1. **Commutative laws**
     $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
  2. **Associative laws**
     $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  3. **Idempotent laws**
     $a \vee a = a$ and $a \wedge a = a$
  4. **Absorption laws**
     $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

- *Be noted that $\vee$ and $\wedge$ does not necessary to be OR and AND. They can be any binary operation which fulfill the following properties*

Lattice: Theorem: Example 1

- Given $(P(\{x,y\}), \preccurlyeq)$
  - Is it a partial order? 1. Reflexive ✓ 2. Antisymmetric ✓ 3. Transitive ✓
  - Is it a lattice? ✓
  - What should be $\vee$ or $\wedge$?
    - $\vee = \cup$, $\wedge = \cap$ ✓ $\preccurlyeq = \subseteq$
    - $\vee = \cap$, $\wedge = \cup$ ✓ $\preccurlyeq = \supseteq$



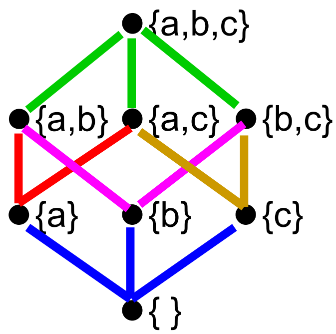


lub: $a \vee b$
glb: $a \wedge b$

1. **Commutative laws**
   $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
2. **Associative laws**
   $a \vee (b \vee c) = (a \vee b) \vee c$ and
   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
3. **Idempotent laws**
   $a \vee a = a$ and $a \wedge a = a$
4. **Absorption laws**
   $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

Lattice: Theorem: Example 2

- Given L as $(P(\{a, b, c\}), \subseteq)$, $\vee = \cup$ and $\wedge = \cap$
- Recall, lub: $a \vee b$
  glb: $a \wedge b$

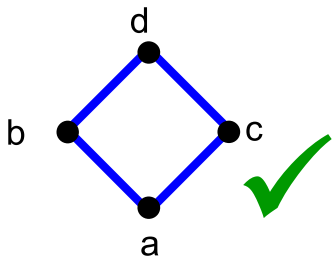


- lub of {a} & {a,b}? $\{a\} \cup \{a,b\} = \{a,b\}$
- lub of {a} & {b,c}? $\{a\} \cup \{b,c\} = \{a,b,c\}$
- glb of {a,b} & {b,c}? $\{a,b\} \cap \{b,c\} = \{b\}$

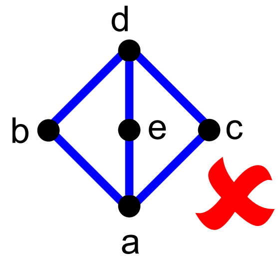
Distributive Lattice

- A lattice $(L, \vee, \wedge)$ is **distributive** if the following identity holds for all $a, b, c \in L$:
  - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
  - $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

- Example,
  - Are they distributive?



$b \wedge (e \vee c) = b \wedge d = b$
$(b \wedge e) \vee (b \wedge c) = a \vee a = a$



Bounded Lattice

- A lattice $(L, \preccurlyeq)$ is called **bounded lattice** if there exist elements $\alpha, \beta \in L$ such that for each $x \in L$, $x \preccurlyeq \alpha$ and $\beta \preccurlyeq x$.
  - α is the largest element of L (denoted by 1)
  - β is the smallest element of L (denoted by 0)
- If a lattice is bounded, then
  - 1 is the lub of the lattice
  - 0 is the glb of the lattice

Complemented Lattice

lub: a ∨ b
glb: a ∧ b

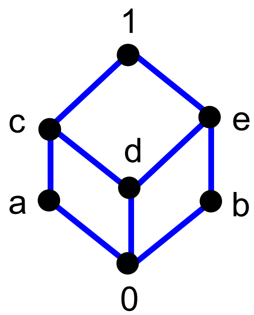
- A bounded lattice (L,≼) is **complemented lattice** if for each x ∈ L, there exists y ∈ L such that x ∨ y = 1 and x ∧ y = 0
  - y is a complement of x (denoted by ¬x)

- In general an element may have more than one complement

Complemented Lattice
Example

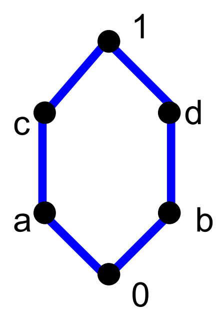
lub: a ∨ b
glb: a ∧ b

y is ¬x if x ∨ y = 1 and x ∧ y = 0



x	¬x
0	**1**
1	**0**
a	**b,e**
b	**a,c**
c	**b**
d	**NO**
e	**a**

Not Complemented Lattice



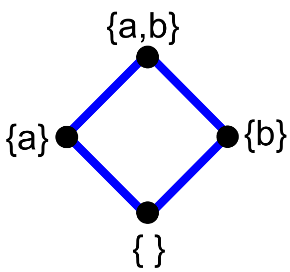
Complemented Lattice

x	¬x
0	**1**
1	**0**
a	**b,d**
b	**a,c**
c	**b,d**
d	**a,c**

Lattice: Principle of Duality

- Any statement that is **true** for lattice remains true when $\preccurlyeq$ is replaced by $\succcurlyeq$ and $\wedge$ and $\vee$ are interchanged throughout the statement.
- Example of dual

$\vee = \cup$
$\wedge = \cap$
$\preccurlyeq = \subseteq$



$\vee = \cap$
$\wedge = \cup$
$\preccurlyeq = \supseteq$