Discrete Mathematic

Chapter 5: Relation

5.4 Closures of Relations

5.5 Equivalence Relations

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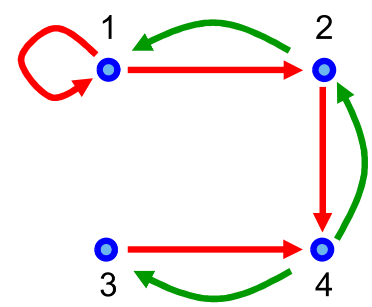
Agenda

- 5.4 Closures of Relations
  - Reflexive Closure
  - Symmetric Closure
  - Transitive Closure
- 5.5 Equivalence Relations
  - Equivalence Relations
  - Equivalence Class
  - Partition

Introduction: Closures

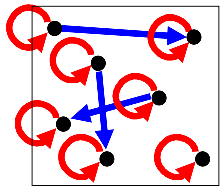
- Is it symmetric?
- How can we produce a symmetric relation containing R that is as small as possible?

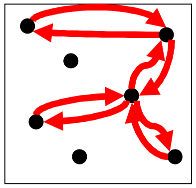


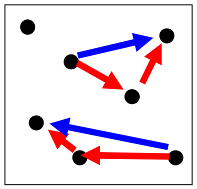
Closure

- Let R be a relation on a set A
- S is called the closure of R with respect to property P if
  - S with property P
  - S is a subset of every relation with property P containing R
    - Minimum terms are added to R to fulfill the requirements of property P

Closure

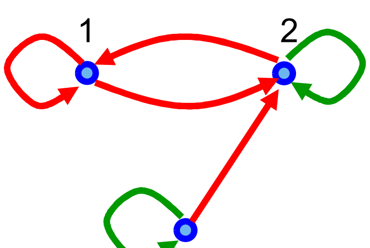






- Reflexive Closure
  - ∀a ((a, a) ∈ R)
- Symmetric Closure
  - ∀a ∀b (((a, b)∈R) → ((b, a)∈R))
- Transitive Closure
  - ∀a∀b∀c (((a,b)∈R∧(b,c)∈R) → ((a,c)∈R))

Closure
Reflexive Closure: Example

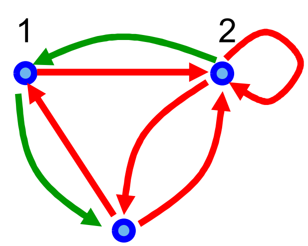


- R = {(1,1), (1,2), (2,1), (3,2)} on the set A = {1, 2, 3}
- R is not reflexive
- How can we produce a reflexive relation containing R that is as small as possible?
  - Add (2,2) and (3,3)
- R' = {(1,1), (1,2), (2,1), (3,2), (2,2), (3,3)}
- R' is reflexive closure of R
  - Any reflexive relation that contains R must contain R'

Closure
Reflexive Closure

- **r(R)** denotes the reflexive closure of R
- How to create a reflexive closure for R?
  - **Graphical view**
    - Add loop for each element
  - **Mathematical View**
    - Let **D** (or *I*) be the **diagonal relation** (equality relation) on R, where D = {(x, x) | x ∈ R}
    - The reflexive closure of R is **R ∪ D**

Closure
Symmetric Closure: Example

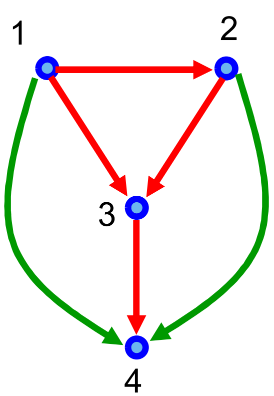


- R = {(1,2), (1,2), (2,2), (2,3), (3,1), (3,2)} on {1, 2, 3}
- R is not symmetric
- How can we produce a symmetric relation containing R that is as small as possible?
  - Add (2,1) and (1,3)
- R' = {(1,2), (1,2), (2,2), (2,3), (3,1), (3,2), (2,1), (1,3)}
- R' is symmetric closure of R
  - Any symmetric relation that contains R must contain R'

Symmetric Closure

- **s(R)** denotes the symmetric closure of R
- How to create a symmetric closure for R?
  - **Graphical view**
    - Add edges in the opposite direction
  - **Mathematical View**
    - Let $\mathbf{R^{-1}}$ be the **inverse** of R, where $R^{-1} = \{(y,x) \mid (x,y) \in R\}$
    - The symmetric closure of R is $\mathbf{R \cup R^{-1}}$
- Theorem: R is symmetric iff $R = R^{-1}$

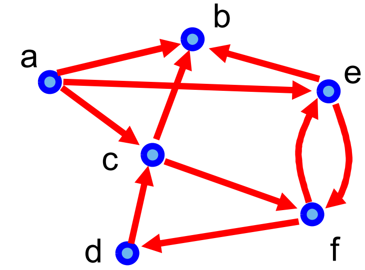
Transitive Closure: Example



- R = {(1,2), (1,3), (2,3), (3,4)} on {1,2,3,4}
- R is not transitive
- How can we produce a transitive relation containing R that is as small as possible?
  - Add (1,4), (2,4)
- R' = {(1,2), (1,3), (2,3), (3,4), (1,4), (2,4)}
- R' is transitive closure of R
  - Any transitive relation that contains R must also contain R'

Transitive Closure

- **t(R)** denotes the transitive closure of R
- How to create a transitive closure for R?
  - **Graphical view**
    - If there is a path from a to b and b to c, add an edge from a to c
    - However, it is not easy
    - Example:



  - **Mathematical View**
    - Transitive Closure of R is R*

Transitive Closure

- The connectivity relation of the relation *R*, denoted ***R****, is the union of R^i, where i = 1,2,3,…

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

- Transitive Closure of R is R*

Closure

Transitive Closure

- **Theorem**
  If $R \subset S$, then $R \circ S \subset S \circ S$
- **Theorem**
  If R is transitive then so is R^n
- **Theorem**
  R is transitive iff $R^n \subseteq R$ for $n > 0$

Closure

Transitive Closure

- Proof: Transitive Closure of R is R^*
  1. R^* is a transitive relation
  2. R^* contains R
  3. R^* is the smallest transitive relation which contains R

Closure

Transitive Closure

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Proof: 1. R^* is a transitive relation

- Suppose (x, y) and (y, z) are in R^*
  Show (x, z) is in R^*
- By definition of R^*, (x, y) is in R^m for some m and (y, z) is in R^n for some n.
- Then (x, z) is in $R^n \circ R^m = R^{m+n}$ which is contained in R^*
- Hence, R^* must be transitive

Proof: 2. R^* contains R

- The proof is obvious by the definition of R^*

Closure

Transitive Closure

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Proof: 3. R^* is the smallest transitive relation which contains R

- Now suppose *S* is any transitive relation that contains R
- Show *S* contains R^*
- Since S is transitive, $S^n \subset S$
- For the power is 2,
  $R^2 = R \circ R \subset S \circ R \subset S \circ S$
- It is true for n, $R^n \subset S^n$
- Therefore $R^n \subset S^n \subset S$ for all n
- Hence S must contain R^* since it must also contain the union of all the powers of R

Theorem:
R is transitive iff $R^n \subseteq R$ for $n > 0$

Theorem:
If $R \subset S$, then $R \circ S \subset S \circ S$

Transitive Closure

- How can we calculate the infinite union?

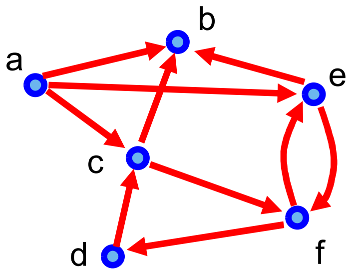
$$R^* = \bigcup_{n=1}^{\propto} R^n$$

Infinity

  - If it is necessary to calculate all R^i?

Transitive Closure

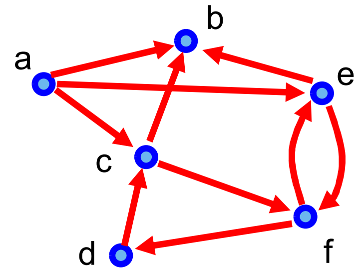
- A path of length n in a digraph G is a sequence of edges $(x_0, x_1),(x_1, x_2),\ldots,(x_{n-1}, x_n)$
- A cycle is a path with starting point (x_0) = end point (x_n)



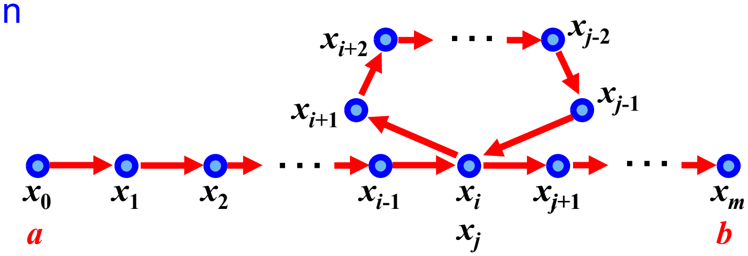
a > e > f > d	Path Length = 3
a > e > b > c	Not a path
c > f > d > c	Cycle Length = 3

Transitive Closure

- Let A be a set with n elements, and let R be a relation on A
- If there is a path from a to b, then the length of this path will not exceed n



- **Proof**
  - Suppose there is a path from a to b in R
  - Let m be the length of the shortest path, which is $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$
  - Assume $m > n$
  - Because n vertices in A and there are m vertices in the path, at least two vertices in the path are equal
  - Suppose that $x_i = x_j$ with $0 \le i < j \le m$
  - There is a path contained a cycle from x_i to itself (x_j)
  - This cycle can be removed to shorten the path
  - Hence, the shortest length must be less than or equal to n



Closure
Transitive Closure

- From the Theorem, we know that R^k for $k > n$ does not contain any edge that does not already appear in the first n powers of R

- Assume R is the relation on set A

$$R^* = \bigcup_{k=1}^{\infty} R^k = \bigcup_{k=1}^{|A|} R^k$$

Closure
Transitive Closure

- **Theorem**
  Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R* is

$$M_{R^*} = M_R^{[1]} \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

$$\text{Remark}: \quad M_{R^k} = M_R^{[k]}$$

$$M_R = M_R^{[1]}$$

Closure
Transitive Closure: Example

- Find the zero-one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

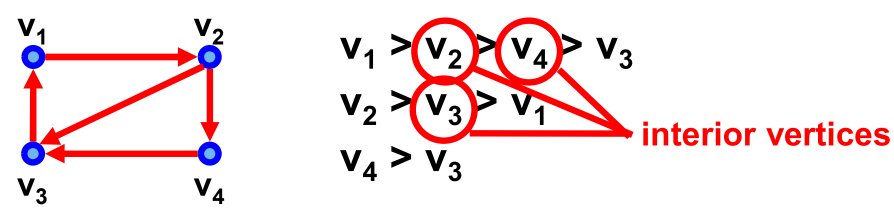
$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Closure: Transitive Closure
Warshall's Algorithm

- Warshall's Algorithm can reduce the complexity of R* calculation
- For the path

  a | $x_1, x_2, \ldots, x_{m-1}$ | b

  the **interior vertices** are $x_1, x_2, \ldots, x_{m-1}$
  - All the vertices of the path except the first and last vertices

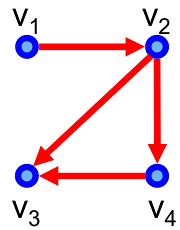


Closure: Transitive Closure

Warshall's Algorithm

- **Warshall's algorithm** is based on the construction of a sequence of zero-one matrices, $W_0, W_1, \ldots, W_n$, where $W_0 = M_R$

$$W_k = \begin{bmatrix} w_{11}(k) & w_{12}(k) & \cdots & \cdots \\ w_{21}(k) & \ddots & & \vdots \\ \vdots & & w_{ij}(k) & \vdots \\ \vdots & \cdots & \cdots & \ddots \end{bmatrix}$$



$w_{14}(0) = 0$

$w_{14}(1) = 0$

$w_{14}(2) = 1$

- $w_{ij}(k) = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$; otherwise is **0**

Closure: Transitive Closure

Warshall's Algorithm

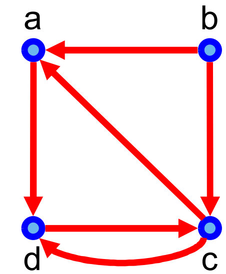
- The $(i,j)^{th}$ entry of M_{R^*} is 1 iff there is a path from v_i to v_j with all the interior vertices in the set $\{v_1, v_2, \ldots, v_n\}$, therefore, $\mathbf{W_n = M_{R^*}}$
- **Algorithm**
  - $W_0 = M_R$
  - For k = 1 … n
    - Update each element in W_k by using:

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$$

Closure: Transitive Closure

Warshall's Algorithm: Example

- Find the matrices W_0, W_1, W_2, W_3 and W_4 for the R shown in the directed graph



- Let v_1=a, v_2=b, v_3=c, v_4=d. W_0 is the matrix of the relation. Hence,

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Closure: Transitive Closure

Warshall's Algorithm: Example

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$$

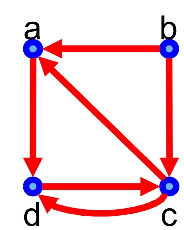
$k = 1$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} i=1 \\ i=2 \\ i=3 \\ i=4 \end{matrix}$$

$k = 2$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_4 is the matrix of the transitive closure



$k = 3$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$k = 4$

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Closure: Theorem

- Let R be binary relation on a nonempty set A
  - r(R) = $R \cup D$
    where **D** is **diagonal relation** = {(x, x) | x $\in$ R}
  - s(R) = $R \cup R^{-1}$
    where R^{-1} is **inverse** = {(y,x) | (x,y) $\in$ R}
  - t(R) = **R***
    where **R*** is **connectivity relation** = $\bigcup_{k=1}^{|A|} R^k$

Closure: Theorem

- Let R be binary relation on a nonempty set A
  - If R is reflexive, r(R) = R
  - If R is symmetric, s(R) = R
  - If R is transitive, t(R) = R

Closure: Theorem

- Let R be binary relation on a nonempty set A
  - If R is reflexive, s(R) and t(R) are reflexive
  - If R is symmetric, t(R) and r(R) are symmetric
  - If R is transitive, r(R) is transitive

Closure: Theorem

- Suppose R is transitive, is s(R) transitive?
- Let R= {(1,2),(3,2)}
- R is transitive
- s(R) = {(1,2), (2,1), (3,2), (2,3)}
- s(R) is not transitive

Closure: Theorem

- Let R be binary relation on a nonempty set A
  - If R is reflexive, s(R) and t(R) are reflexive
  - If R is symmetric, t(R) and r(R) are symmetric
  - If R is transitive, r(R) is transitive

  - r(s(R)) = s(r(R))? ✓
  - r(t(R)) = t(r(R))? ✓
  - s(t(R)) = t(s(R))? ✗

Closure: Theorem

- Proof r(s(R)) = s(r(R))

- s(r(R)) = s(R U D) where **D** = $\{(x, x) \mid x \in R\}$

  = $(R \cup D) \cup (R \cup D)^{-1}$

  = $(R \cup D) \cup (R^{-1} \cup D^{-1})$

  = $(R \cup R^{-1}) \cup (D \cup D^{-1})$

  = s(R) U D

  = r(s(R))

Closure: Theorem

- Do the closure operations distribute
  - over the set operations?
  - over inverse?
  - over complement?
  - over set inclusion?

  - Example:
    - $t(R_1 - R_2) = t(R_1) - t(R_2)$?
    - $r(R^{-1}) = (r(R))^{-1}$?

Equivalence

- What is Equivalence?

=

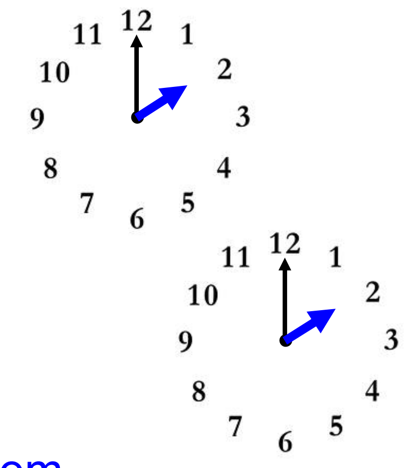
- What properties the equivalence should have?

Reflexive ~~Irreflexive~~ Transitive

Symmetric ~~Asymmetric~~ ~~Antisymmetric~~

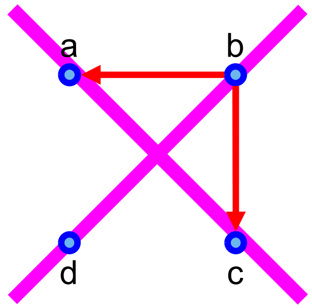
Equivalence

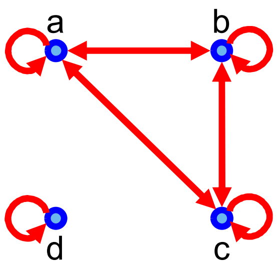
- How to represent "**2**" in clock system?
- How to represent "**14**" in clock system?
- Clock System is Arithmetic modulo 12
- "2", "14", "26", "38"… are equivalence in clock system



Equivalence Relations

- **Definition**
  A relation R on a set A is an **equivalence relation** iff R is reflexive, symmetric and transitive





Equivalence Relation

Equivalence Relations

Example 1

- Suppose that **R** is the relation on the set of strings of English letters such that **aRb iff g(a)=g(b)**, where g(x) is the length of the string x.
  Is R an equivalence relation?
- **Reflexive**
  - Since g(a)=g(a), it follows that aRa whenever a is a string
- **Symmetric**
  - Let aRb, so g(a)=g(b), bRa. Therefore, g(b)=g(a)
- **Transitive**
  - Let aRb and bRc, then g(a)=g(b) and g(b)=g(c), so aRc
- Consequently, R is an equivalent relation

Equivalence Relations

Example 2

b = x · m + a where x is an integer x = (b-a) / m

- Definition of **Congruence**

  a ≡ b (mod m)

  a is <u>congruent</u> to b <u>modulo</u> m if m divides a-b

- Let m be a positive integer greater than 1. Show that the relation R = { (a,b) | a ≡ b (mod m) } is an equivalence relation on the set of integers

Equivalence Relations
Example 2

R = { (a,b) | a ≡ b (mod m) }
b = x · m + a
where x is an integer
x = (b-a) / m

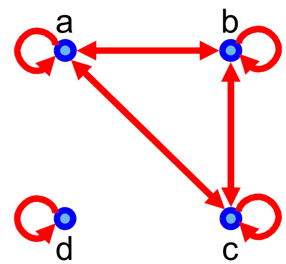
- **Reflexive**
  - a – a = 0 is divisible by m, hence, a ≡ a (mod m)
- **Symmetric**
  - Suppose that (a, b)∈R, so x = (b-a)/m, where x is an integer
  - (-x) = (a-b) / m, -x is also an integer, (b, a)∈R
- **Transitive**
  - Suppose that (a,b) ∈ R and (b,c) ∈ R
  - xm = (b-a) and ym = (c-b), x and y are integers
  - a-c = xm+ym = (x+y)m, x+y is also an integer
  - Thus, (a, c)∈R

Equivalence Relations
Example 3

- Show that the "divides" relation on the set of positive integers is an equivalence relation.
- “Divide” relation is not symmetric
  - E.g., 2 divide 4 but 4 does not divide 2
- It is not an equivalence relation

Equivalence

- Two elements a and b that are related by an equivalence relation are called **equivalent**
- Notation: a ~ b



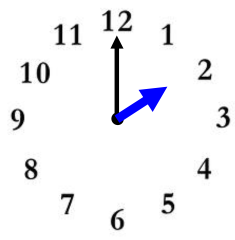
a ~ a	c ~ a
a ~ b	c ~ b
a ~ c	c ~ c
b ~ a	d ~ d
b ~ b	
b ~ c	

Equivalence: Examples

- R is the relation on the set of strings of English letters, where aRb iff g(a)=g(b) and g(x) is the length of the string x
  - “Peter” ~ “Susan”
  - “Ann” ~ “May”
- R = { (a,b) | a ≡ b (mod m) } on the set of integers
  - For m = 7, 5 ~ 12
  - For m = 12, 14 ~ 2

Equivalence Classes

- **Definition**
  Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a
- Example (clock system)
  - "2", "14", "26", "38"… are equivalence
  - Therefore, they form an equivalence class



Equivalence Classes

- The equivalence class of a with respect to R is denoted by $[a]_R$

$$[a]_R = \{s \mid (a,s) \in R\}$$

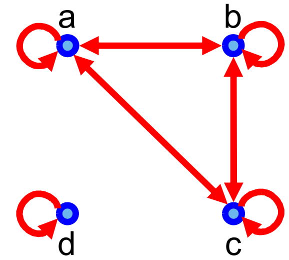
- If $b \in [a]_R$, b is called a representative of this equivalence class

Equivalence Classes

Example 1

$[a]_R = \{s \mid (a,s) \in R\}$

- Equivalence class of
  - [a] = {a, b, c}
  - [b] = {a, b, c}
  - [c] = {a, b, c}
  - [a] = [b] = [c]
  - [d] = {d}



Equivalence Classes

Example 2

$[a]_R = \{s \mid (a,s) \in R\}$

- R = { (a,b) | a ≡ b (mod m) } is an equivalence relation on the set of integers, where m be a positive integer greater than 1
  - Let m = 5
    - R = { (a,b) | a ≡ b (mod 5) }
    - [0] = { **…, -10, -5, 0, 5, 10, …** }
    - [1] = { **…, -9, -4, 1, 6, 11, …** }
    - [a] = { **…, a-10, a-5, a, a+5, a+10, …** }
  - General Case, for any m,
    - [a] = { **…, a-2m, a-m, a, a+m, a+2m, …** }

Equivalence Classes

Example 3

$[a]_R = \{s \mid (a,s) \in R\}$

- R is the relation on the set of strings of English letters, where aRb iff g(a)=g(b) and g(x) is the length of the string x
  - [e] = { **a, b, c, …, z** }
  - [Susan] = { **happy, email, …** }
  - For any a,
    [a] = the set of all strings of the same length as a

Equivalence Classes

Theorem

- Let R be an equivalence relation on a nonempty set A. The following statements are equivalent:
  1. aRb
  2. [a] = [b]
  3. $[a] \cap [b] \neq \emptyset$

Equivalence Classes

Theorem: Proof

1. aRb
2. [a] = [b]
3. $[a] \cap [b] \neq \emptyset$

- **Show (1) implies (2)**
  - Assume that aRb
  - Suppose $c \in [a]$. Then aRc
  - As aRb and R is symmetric, we have bRa
  - Furthermore, since R is transitive and bRa and aRc, it follows that bRc
  - Hence, $c \in [b]$
  - This shows that $[a] \subseteq [b]$
  - The proof that $[b] \subseteq [a]$ is similar.
- **Show (2) implies (3)**
  - Assume that [a] = [b]
  - It follows that $[a] \cap [b] \neq \emptyset$ since [a] is nonempty

Equivalence Classes

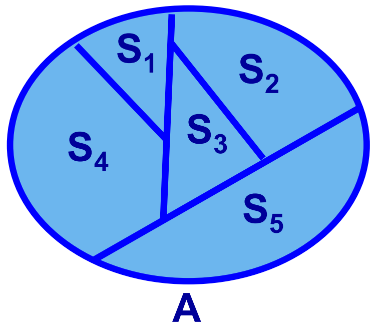
Theorem: Proof

1. aRb
2. [a] = [b]
3. $[a] \cap [b] \neq \emptyset$

- **Show that (3) implies (1)**
  - Suppose that $[a] \cap [b] \neq \emptyset$
  - Then there is an element $c \in [a]$ and $c \in [b]$
  - In other words, aRc and bRc
  - By the symmetric property, cRb
  - Then by transitive, since aRc and cRb, we have aRb.
- Since (1) implies (2),(2) implies (3), and (3) implies (1), the three statements are equivalent.

Equivalence Classes & Partitions

- **Definition**
  Let $S_1, S_2, ..., S_n$ be a collection of subsets of A. The collection forms **a partition of A** if the subsets are
  1. Nonempty
     $S_i \neq \varnothing$
  2. Disjoint
     $S_i \cap S_j = \varnothing$ if $i \neq j$
  3. Exhaust A
     $\bigcup_{i=1}^{n} S_i = A$



Theorem 1

- Let R be an equivalent relation on a set A. Then the equivalence classes of R form a partition of A
- Conversely, given a partition $\{S_i \mid i \in C\}$ of the set A, there is an equivalence relation R that has the sets S_i, where $i \in C$, as its equivalence classes

Theorem 2

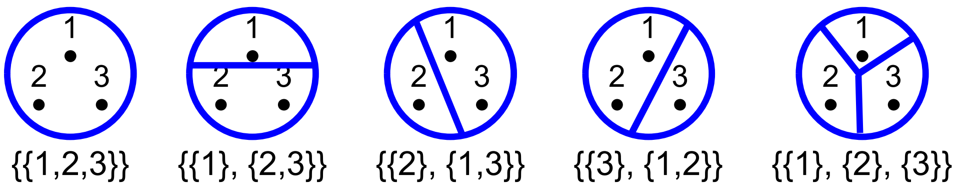
- Equivalence classes of an equivalence relation R partition the set A into disjoint nonempty subsets whose union is entire set
- This partition is denoted **A/R** and called
  - Quotient set, or
  - Partition of A induced by R, or
  - A modulo R
- The partition is a set of equivalence classes whose union is the entire set

Example 1

- What are the sets in the partition of the integers arising from congruence modulo 4?
- There are four congruence classes, corresponding to $[0]_4$, $[1]_4$, $[2]_4$ and $[3]_4$.
  - $\mathbf{[0]_4}$ ={…,-8,-4,0,4,8,…}
  - $\mathbf{[1]_4}$ ={…,-7,-3,1,5,9,…}
  - $\mathbf{[2]_4}$ ={…,-6,-2,2,6,10,…}
  - $\mathbf{[3]_4}$ ={…,-5,-1,3,7,11,…}
- The quotient set: $Z/R = \{ [0]_4, [1]_4, [2]_4, [3]_4 \}$

Example 2

- Let A={1, 2, 3}, give all the possible partitions on A.



Example 3

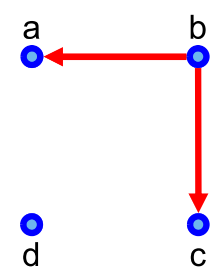
- List the ordered pairs in the equivalence relation R produced by the partition A_1 = {1, 2, 3}, A_2 = {4, 5}, and A_3 = {6} of S = {1, 2, 3, 4, 5, 6}
- For A_1: **(1,2), (1,3), (2,3), (2,1), (3,1), (3,2), (1,1), (2,2), (3,3)**
- For A_2: **(4,5), (5,4), (4,4), (5,5)**
- For A_3: **(6,6)**

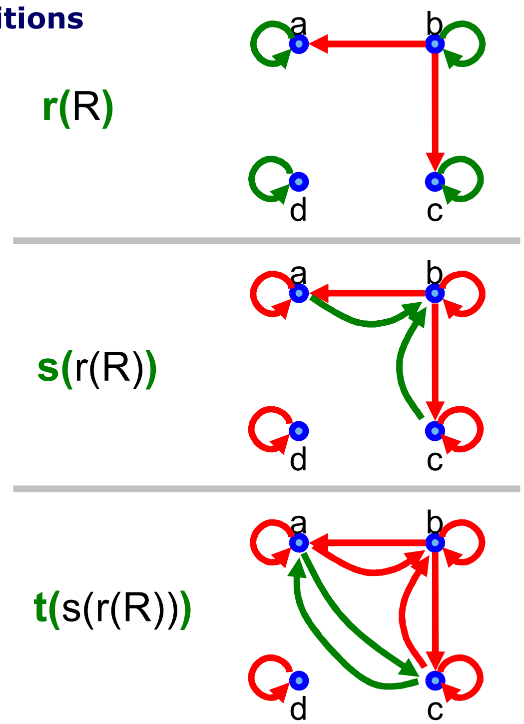
Theorem 3

- Let R be a relation on A. Reflexive, Symmetric, Transitive closure of R, tsr(R) = t(s(r(R))), is an equivalence relation on A, called the **equivalence relation induced by R**

Theorem 3

- t(s(r(R)))
  1. **r**(R)
  2. **s(**r(R)**)**
  3. **t(**s(r(R))**)**





Theorem 3: Proof

- Proof: tsr(R) is an equivalence relation
  - **Reflexive**
    - When constructing r(R), a loop is added at every element in A, therefore, tsr(R) must be reflexive
  - **Symmetric**
    - If there is an edge (x, y) then the symmetric closure of r(R) ensures there is an edge (y, x)

Theorem 3

- **Transitive**
  - When we construct the transitive closure of sr(R), an edge (a, c) is added if (a, b) and (b, c)
  - tsr(R) must be transitive
  - As sr(R) is symmetric, if (a, b) and (b, c) in sr(R), (b, a) and (c, b) are also in sr(R). Therefore, another edge (c, a) is also added
  - It guarantees that tsr(R) is symmetric