Chapter 2: Set Theory

2.3 Functions

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Agenda

- Functions
- One-to-One Functions
- Onto Functions
- Increase/Decrease Functions
- Inverse Functions
- Composition of Functions
- Graphs of Functions
- Floor and Ceiling Functions
- Factorial Functions

Functions

- The concept of a **function** is extremely important in mathematics and computer science
  - Sometimes, a function is called **mapping** or **transformation**

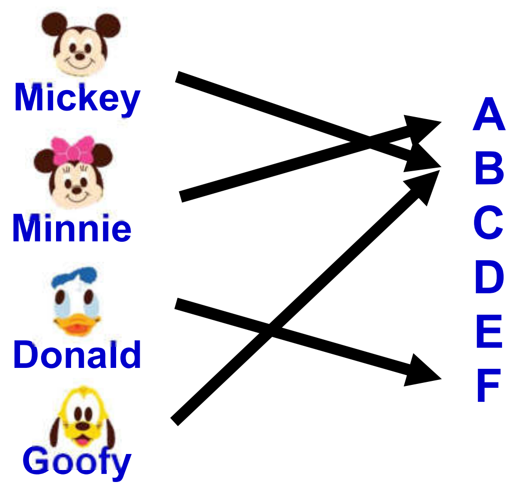
Functions

- In Mathematic…
  - $f(x) = x^2$
  - $g(x, y) = x + y$
- In Programming…
  - float square (float x) {return x^2}
  - float sum (float x, float y) {return x+y}

Input ⇨ **Process** ⇨ **Output**

Functions

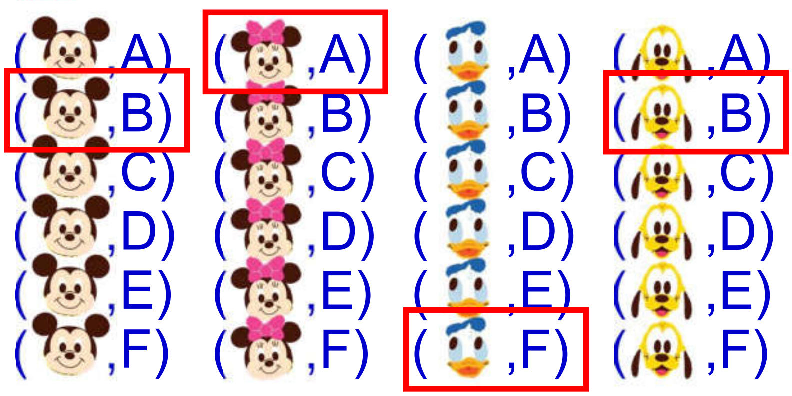
- Let **A** and **B** be nonempty sets
  Function ***f*** from **A** to **B** is an assignment of exactly one element of **B** to each element of **A**
- Denoted by ***f*** **(a) = b**
  b is the unique element of B assigned by the function ***f*** to the element **a** of A
- If ***f*** is a function from **A** to **B**, we write ***f*** **: A → B**

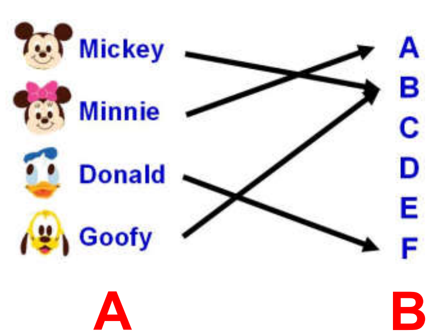


Functions

- A function ***f*** : A → B can also be defined in terms of a relation from A to B
  - A **relation** from A to B is a **subset** of A x B
  - By **defining** using a **relation**, a **function** from A to B contains **unique** ordered pair (a, b) for **every** element a ∈ A

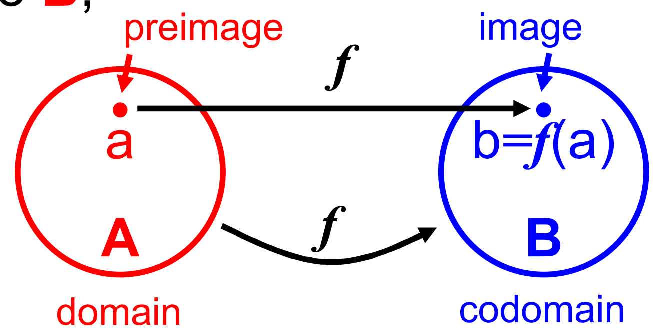
AxB



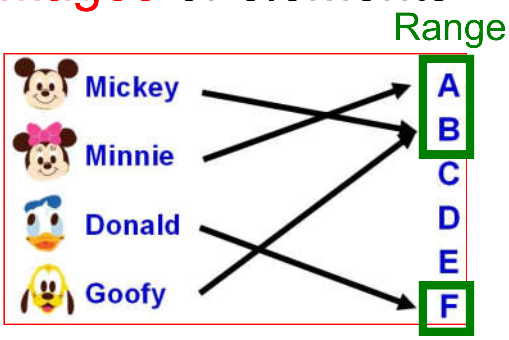


Functions

- If f is a function from **A** to **B**,
  - **A** is the **domain** of f
  - **B** is the **codomain** of f
- If f**(a) = b**,
  - **b** is the **image** of **a**
  - **a** is a **preimage** of **b**



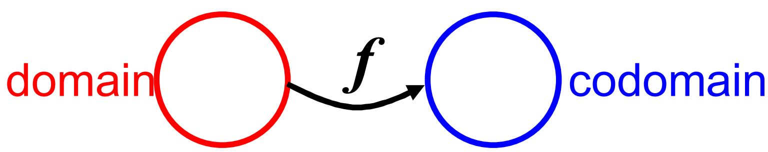
- The **range** of f is the set of all images of elements of A
- If f is a function from A to B, we say that f **maps** A to B



Functions

Input ⇨ **Process** ⇨ **Output**

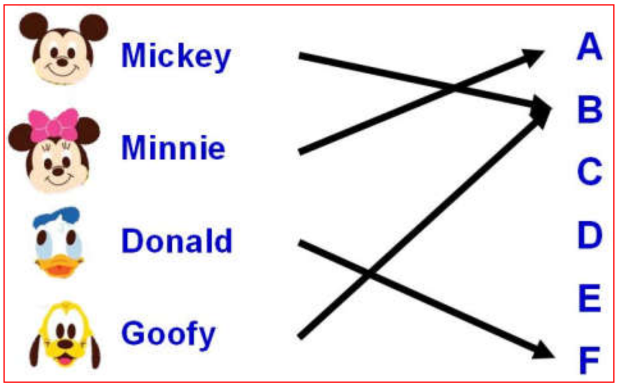
- A **function is defined** by specifying
  1. **Domain**
  2. **Codomain**
  3. **Mapping** of elements of the domain to elements in the codomain



- Two functions are **equal** when these three things are the same
- Therefore, either these three things is changed, we have a new function

Functions: Example 1

- What are the domain, codomain, and range of the function that assigns grades to students?
- Let G be the function that assigns a grade to a student, E.g. G(Mickey) = B
- Domain of G = {Mickey, Minnie, Donald, Goofy}
- Codomain of G = {A, B, C, D, E, F}
- Range of G = {A, B, F}



Functions: Example 2

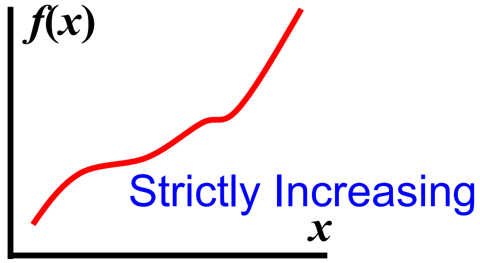
- Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. What are the domain, codomain and range?
- For example, f(11010) = 10
- Domain is the set of
  all bit strings of length 2 or greater
- Codomain is set {00, 01, 10, 11}
- Range is set {00, 01, 10, 11}

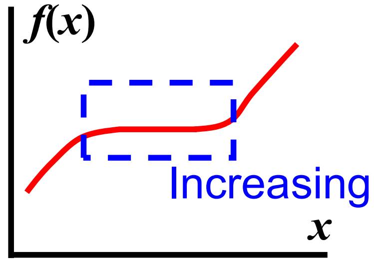
Function Example 3

In/Decrease Functions

- Function f whose domain and codomain are subsets of the set of real numbers is called
  - Increasing if $f(x) \leq f(y)$
  - Decreasing if $f(x) \geq f(y)$
  - Strictly Increasing if $f(x) < f(y)$
  - Strictly Decreasing if $f(x) > f(y)$

  whenever $x < y$ and x and y are in the domain of f





Function Example 3

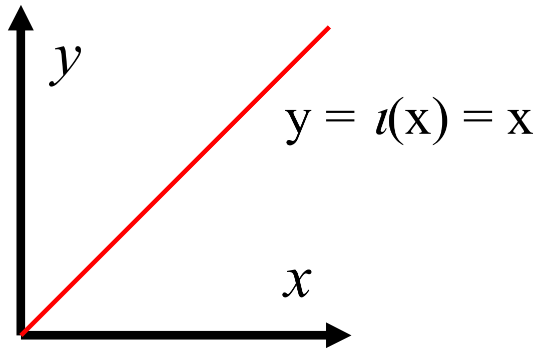
In/Decrease Functions

- A function f is
  - Increasing if $\forall x \forall y\ (x<y \rightarrow f(x) \leq f(y))$
  - Decreasing if $\forall x \forall y\ (x<y \rightarrow f(x) \geq f(y))$
  - Strictly Increasing if $\forall x \forall y\ (x<y \rightarrow f(x) < f(y))$
  - Strictly Decreasing if $\forall x \forall y\ (x<y \rightarrow f(x) > f(y))$

  where the universe of discourse is the domain of f

Function Example 4
Identity Function

- Let A be a set. The **identity function** on A is the function $\iota_A : A \rightarrow A$, where $\iota_A(x) = x$, for all $x \in A$ (Note that ι is the Greek letter iota)
  - Assigns each element to itself
  - Domain = Codomain



Function Example 5
Floor and Ceiling Function

- Let x be a real number
  - **Floor Function**
    - Rounds x down to the closest integer less than or equal to x
    - Notation: $\lfloor \mathbf{x} \rfloor$
    - Sometimes call *greatest integer function* and denoted by **[x]**
  - **Ceiling Functions**
    - Rounds x up to the closest integer greater than or equal to x
    - Notation: $\lceil \mathbf{x} \rceil$

Function Example 5

Floor and Ceiling Function

- Example
  - $\lfloor 0.5 \rfloor =$ **0**
  - $\lceil 0.5 \rceil =$ **1**
  - $\lfloor -0.5 \rfloor =$ **-1**
  - $\lceil -0.5 \rceil =$ **0**
  - $\lfloor 3.1 \rfloor =$ **3**
  - $\lceil 3.1 \rceil =$ **4**
  - $\lfloor 7 \rfloor =$ **7**
  - $\lceil 7 \rceil =$ **7**

- **Floor Function ($\lfloor x \rfloor$)**
  Rounds x down to the closest integer less than or equal to x
- **Ceiling Functions ($\lceil x \rceil$)**
  Rounds x up to the closest integer greater than or equal to x

Function Operation

- Let f_1 and f_2 be **functions** from **A** to **R**
  $f_1 + f_2$ and $f_1 f_2$ are also **functions** from **A** to **R**
  defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

- Functions $f_1 + f_2$ and $f_1 f_2$ have been defined by specifying their values at x in terms of the values of f_1 and f_2 at x

Function Operation: Example

- Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$
- What are the functions $f_1 + f_2$ and $f_1 f_2$?

$(f_1 + f_2)(x)$	$(f_1 f_2)(x)$
$= f_1(x) + f_2(x)$	$= f_1(x) f_2(x)$
$= x^2 + (x - x^2)$	$= x^2 (x - x^2)$
$= x$	$= x^3 - x^4$

Functions: Programming

- **Domain** and **codomain** of functions are often specified in programming languages
- Example,
  - Pascal : function **floor** (x: **real**) : **integer**
  - Java : **int** **floor** (**float** x) {...}
  - C++: **int** **floor** (**float** x) {…}
  - Domain is the set of real numbers
  - Codomain is the set of integers

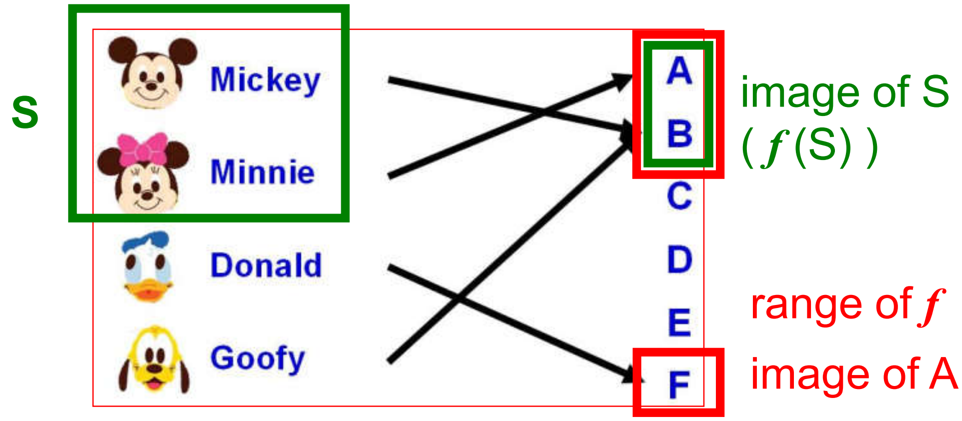
Images of Subset

- Image of S is denoted by $f(S)$, (S is a set)

$$f(S) = \{ t \mid \exists s \in S \ (t = f(s))\} \text{ , or}$$

$$f(S) = \{f(s) \mid s \in S\}$$

- The image of S under the function f is the subset of B that consists of the images of the elements of S



Images of Subset

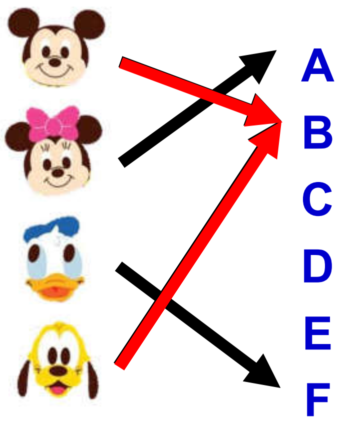
- $f(S)$ may be ambiguous:
  - A set (image of S) $(\{f(s) \mid s \in S\})$
  - Function f for the set S
    (input of a function is a set)
  - We assume $f(S)$ is a set in this course

Images of Subset: Example

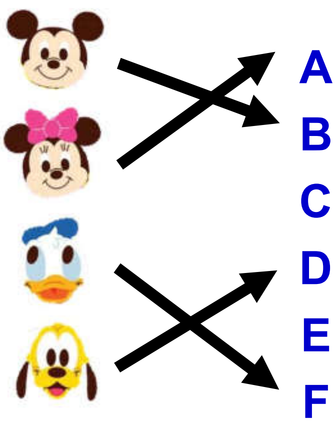
- Let
  A = {a, b, c, d, e} and B = {1, 2, 3, 4} with
  f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, and f(e) = 1
- Given S = {b, c, d}, what is the image of S?
- Image is the set f(S) = {1, 4}

One-to-One Functions

- Function f is said to be **one-to-one**, or **injective**, if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f
- One-to-one functions never assign the same value to two different domain elements



Not one-to-one
(Many-to-one)



One-to-one

One-to-One Functions

- By taking the contrapositive of the definition, a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$
- We can express that f is one-to-one using quantifiers as

$$\forall a\ \forall b\ (f(a) = f(b) \rightarrow a = b) \quad \text{or}$$

$$\forall a\ \forall b\ (a \neq b \rightarrow f(a) \neq f(b)),$$

where the universe of discourse is the domain of the function

One-to-One Functions
Example 1

- Determine if the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one

- Suppose $f(n) = f(m)$
- $n+1 = m+1$, therefore $n = m$
- $f(x)$ is one-to-one

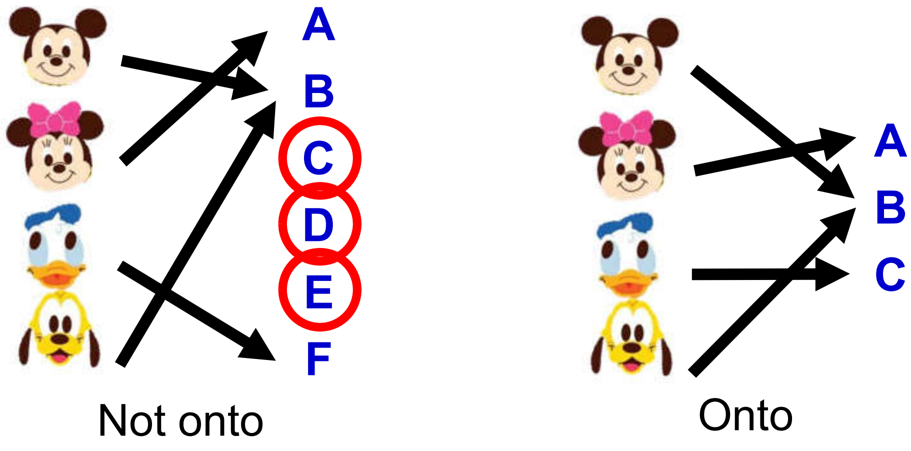
Example 2

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one

- Suppose $f(n) = f(m)$
- $n^2 = m^2$
  - n may be equal to $-m$
  - $n^2 = m^2$ does not imply $n = m$
- $f(x)$ is not one-to-one

Onto Functions

- Function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$
- A function f is onto if $\forall y \, \exists x \, (f(x) = y)$, where
  - Domain for x is the domain of the function
  - Domain for y is the codomain of the function

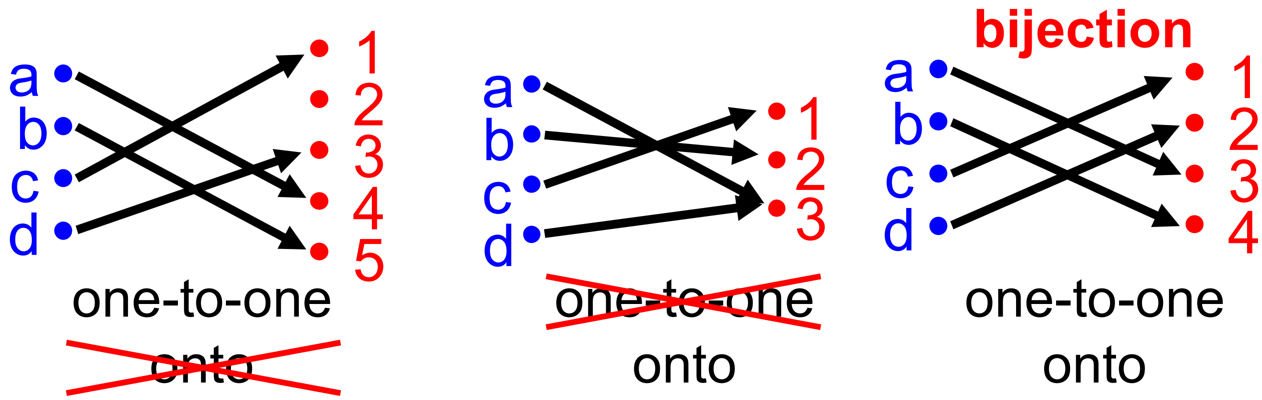


Example

- Example 1
  - Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
  - There is no integer x with $x^2 = -1$, therefore, not onto
- Example 2
  - Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?
  - Yes

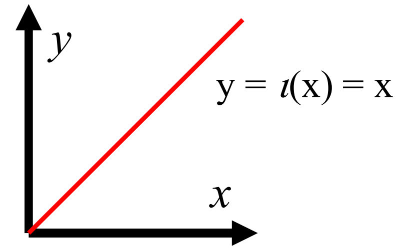
One-to-one and Onto Functions

- The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto
- Suppose that f is a function from a set A to itself
  - If A is **finite**, then f is one-to-one if and only if it is onto
  - If A is **infinite**, this is not necessarily the case
    - We will discuss it later



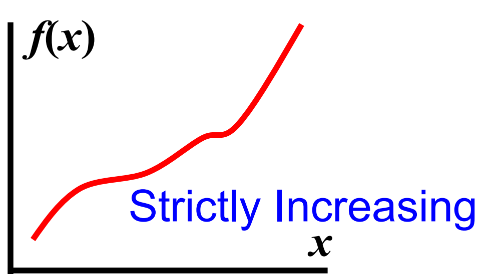
Example 1

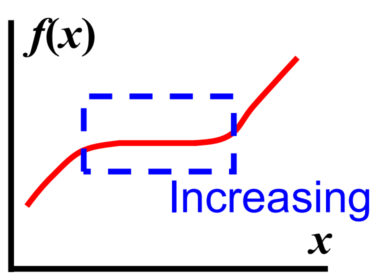
- The **identity function** on A is the function $\iota_A : A \to A$, where $\iota_A(x) = x$, for all $x \in A$
  - Assigns each element to itself
- One-to-one? ✓
- Onto? ✓
- So it is a bijection



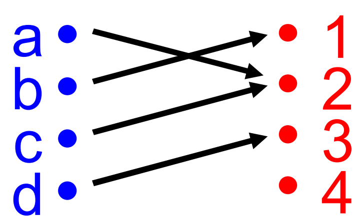
Example 2

- Either strictly increasing or strictly decreasing function must be one-to-one
- Either increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not necessarily one-to-one



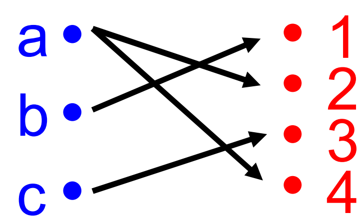


Example 3



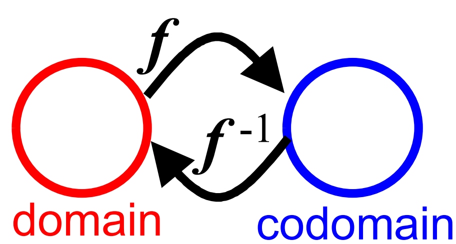
Not one-to-one

Not onto

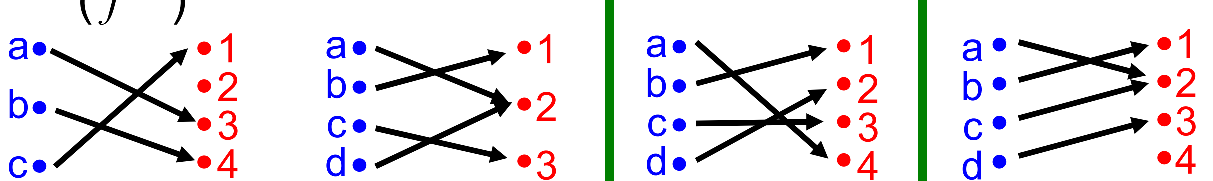


Not a function

Inverse Functions



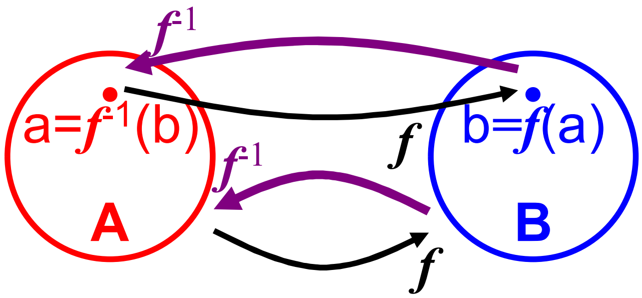
- Recall, the definition of a function
  - Let **A** and **B** be nonempty sets
    Function *f* from **A** to **B** is an assignment of exactly one element of **B** to each element of **A**
    - Every element (a) in domain (A) has only one image (f(a))
- **Inverse function f^{-1}** (reverse processing)
  - f^{-1} : B $\rightarrow$ A
  - Every element (b) in domain (B) has only one image (f^{-1})

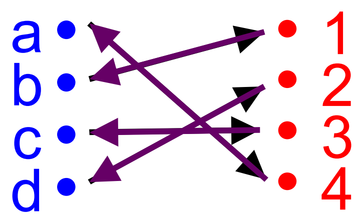


One-to-one Not onto | Not one-to-one Onto | One-to-one Onto | Not one-to-one Not onto

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B, $f(a) = b$
- **Inverse function** of f, denoted by f^{-1}, is the function that assigns to an element b belonging to B the unique element a in A,
  $f^{-1}(b) = a$ when $f(a) = b$



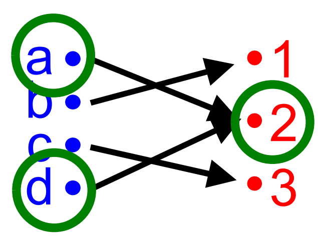


- Be sure not to confuse the function f^{-1} with the function $1 / f$

Inverse Functions

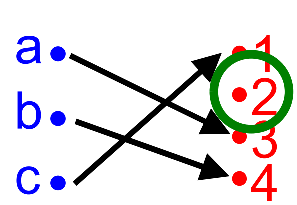
- If a function f is **not** a one-to-one correspondence, an inverse function of f cannot be defined
  - **f is not one-to-one**

    Some b in the codomain is the image of more than one element a



  - **f is not onto**

    Some b in the codomain is the image of no element a



Inverse Functions

- A function having an inverse function is called **invertible**
  - Therefore, a function is not invertible if it is not a one-to-one correspondence

Example 1

- Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

- Because $f(-2) = f(2) = 4$, f is not one-to-one.
- If an inverse function were defined, it would have to assign two elements to 4
- Hence, f is not invertible

Example 2

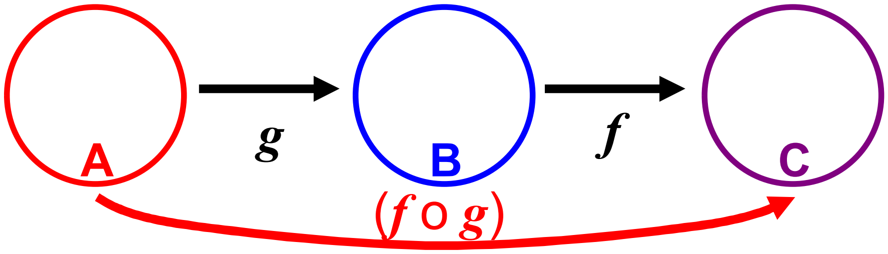
- Show that if we restrict the function $f(x) = x^2$ to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible

- **One-to-One Function Proof**
  - If $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$
  - This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$
  - Because both x and y are nonnegative, we must have $x = y$. It is **one-to-one**
- **Onto Function Proof**
  - The codomain is the set of all nonnegative real numbers, so each nonnegative real number has a square root. It is onto
- Therefore, **f is invertible**

Composition of Function

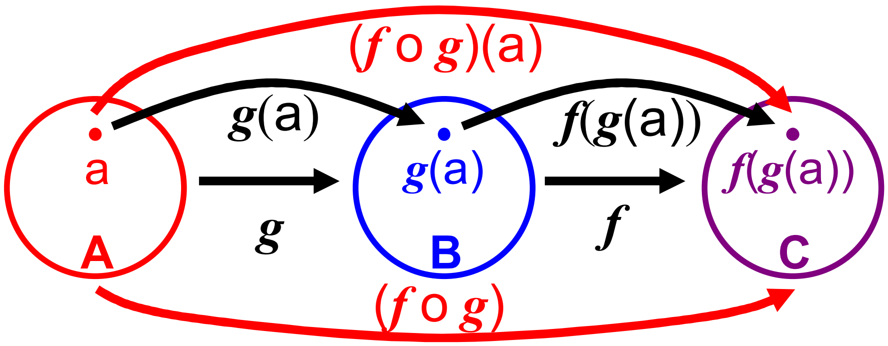
- Let
  - g be a function from the set A to the set B
  - f be a function from the set B to the set C
- The composition of the functions f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(\, g(a)\,)$$



Composition of Function

- To find $(f \circ g)(a) = f(g(a))$
  - the function g is applied to a to obtain $g(a)$
  - the function f is applied to the result $g(a)$ to obtain $f(g(a))$
- Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f



Composition of Function
Example

- Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$
- What is the composition of f and g?
  What is the composition of g and f?
- Both the compositions $f \circ g$ and $g \circ f$ are defined

$(f \circ g)(x) = f(g(x))$
$= f(3x + 2)$
$= 2(3x + 2) + 3$
$= 6x + 7$

$(g \circ f)(x) = g(f(x))$
$= g(2x + 3)$
$= 3(2x + 3) + 2$
$= 6x + 11$

Composition of Function

- The commutative law does not hold for the composition of functions

$$f \circ g \neq g \circ f$$

- It is associative :

$$f \circ (g \circ h) = (f \circ g) \circ h$$

Composition of Function

- Let f is a one-to-one correspondence between A and B, then

$$f^{-1} \circ f = \iota_A \text{ and } f \circ f^{-1} = \iota_B$$

  where ι_A & ι_B are identity functions on sets A & B respectively
- Such that:
  - $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$
  - $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$

Graphs of Functions

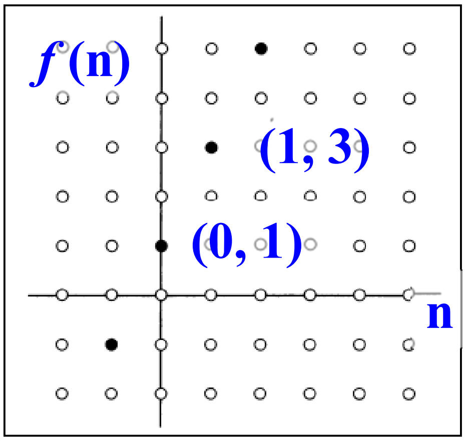
- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs

$$\{(a, b) \mid a \in A \text{ and } f(a) = b\}$$

- It is often displayed pictorially to aid in understanding the behavior of the function

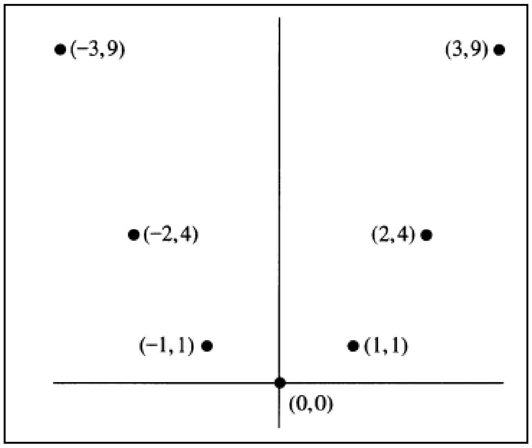
Graphs of Functions
Example 1

- Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers
- The graph of f is the set of ordered pairs of the form $(n, f(n)) = (n, 2n + 1)$, where n is an integer



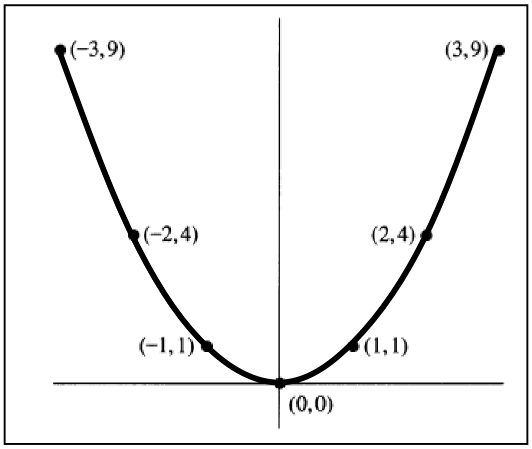
Example 2

- Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers
- The graph of f is the set of ordered pairs of the form (x, x^2), where x is an integer



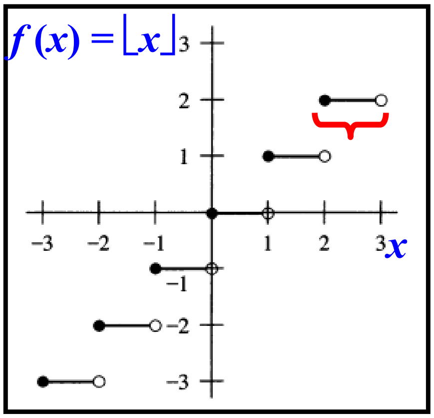
Example 3

- Display the graph of the function $f(x) = x^2$ from the set of real to the set of real
- The graph of f is the set of ordered pairs of the form (x, x^2), where x is an real (infinite Set)



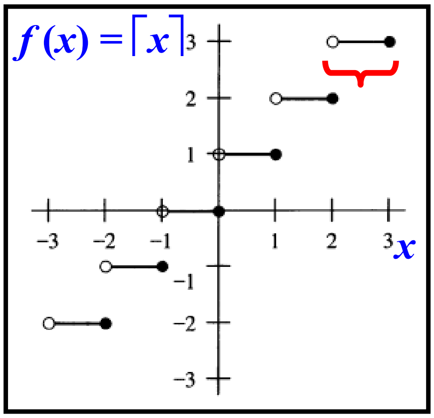
Example 4

- The graph of the floor function $\lfloor x \rfloor$



[n, n + 1)

- The graph of the ceiling function $\lceil x \rceil$



(n, n + 1]