Chapter 1: Logic and Proof

1.5
Rules of Inference
1.6
Introduction to Proofs

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Agenda

- Rules of Inference
- Rules of Inference for Quantifiers

Recall…

- John is a cop. John knows first aid. Therefore, all cops know first aid

Recall…

- Some students work hard to study. Some students fail in examination. So, some work hard students fail in examination.

Argument

p: It rains
q: The floor is wet

Premise / Hypothesis { If it rains, the floor is wet | p
It rains | $p \rightarrow q$

Conclusion { ∴ The floor is wet | ∴ q

therefore

Argument | **Argument Form**

- **Argument** in propositional logic is a sequence of propositions
  - **Premises / Hypothesis:** All except the final proposition
  - **Conclusion:** The final proposition
- **Argument form** represents the argument by variables

Argument: Valid?

- Given an argument, where
  - $p_1, p_2, \ldots, p_n$ be the premises
  - q be the conclusion
- **The argument is valid** when $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology
  - When all premises are true, the conclusion should be true
  - When not all premises are true, the conclusion can be either true or false

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Focus on this case
Check if it happens

Argument

- Example:

Argument is valid

$p \rightarrow q$	If it rains, the floor is wet
p	It rains
q $\therefore$	The floor is wet

$$(\ p \wedge (p \rightarrow q)\ ) \rightarrow q$$

Tautology

Need to check if the conclusion is true or not

Must be true

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rules of Inference

- How to show an argument is valid?
  - **Truth Table**
    - May be tedious when the number of variables is large
  - **Rules of Inference**
    - Firstly establish the validity of some relatively simple argument forms, called **rules of inference**
    - These rules of inference can be used as building blocks to construct more complicated valid argument forms

Rules of Inference

- **Modus Ponens**
  - Affirm by affirming

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore\ q \end{array}$$

- **Modus Tollens**
  - Deny by denying

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore\ \neg p \end{array}$$

Rules of Inference

- **Hypothetical Syllogism**

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

- **Disjunctive Syllogism**

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Rules of Inference

- **Addition**

$$\frac{p}{\therefore\ p \vee q}$$

- **Simplification**

$$\frac{p \wedge q}{\therefore\ p}$$

- **Conjunction**

$$\frac{\begin{matrix} p \\ q \end{matrix}}{\therefore\ p \wedge q}$$

Rules of Inference

- **Resolution**

p = T	p = F
q = T/F	q = T
r = T	r = T/F

$$\frac{\begin{matrix} p \vee q \\ \neg p \vee r \end{matrix}}{\therefore\ q \vee r}$$

- Example
  - I go to swim or I play tennis
  - I do not go to swim or I play football
  - Therefore, I play tennis or I play football

Rules of Inference ($\rightarrow$)

Modus Ponens	$((p \rightarrow q) \wedge (p)) \rightarrow \mathbf{q}$
Modus Tollens	$((\neg q) \wedge (p \rightarrow q)) \rightarrow \neg \mathbf{p}$
Hypothetical Syllogism	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (\mathbf{p} \rightarrow \mathbf{r})$
Disjunctive Syllogism	$((p \vee q) \wedge (\neg p)) \rightarrow \mathbf{q}$
Addition	$(p) \rightarrow \mathbf{p} \vee \mathbf{q}$
Simplification	$((p) \wedge (q)) \rightarrow \mathbf{p}$
Conjunction	$((p) \wedge (q)) \rightarrow (\mathbf{p} \wedge \mathbf{q})$
Resolution	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (\mathbf{q} \vee \mathbf{r})$

Rules of Equivalence ($\leftrightarrow$)

- Recall…

Identify Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Negation Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Comparison between Inference and Equivalence

- **Inference ($p \to q$)**
  - Meaning:
    If p, then q
  - $p \to q$ does not mean $q \to p$
  - Either inference or equivalence rules can be used
  - $p \leftrightarrow q$ implies $p \to q$
  - $\Rightarrow$ is used in proof
- **Equivalence ($p \leftrightarrow q$)**
  - Meaning:
    p is equal to q
  - $p \leftrightarrow q$ mean $q \leftrightarrow p$
  - Only equivalence rules can be used
  - $p \leftrightarrow q$ can be proved by showing $p \to q$ and $q \to p$
  - $\Leftrightarrow$ is used in proof
- Equivalence ($\leftrightarrow$) is a **more restrictive** relation than Inference ($\to$)

Using Rules of Inference

- Example 1:
  - **Given:**
    - It is not sunny this afternoon and it is colder than yesterday.
    - We will go swimming only if it is sunny
    - If we do not go swimming, then we will take a canoe trip
    - If we take a canoe trip, then we will be home by sunset
  - Can these propositions lead to the **conclusion** "We will be home by sunset" ?

Let		
	p:	It is sunny this afternoon
	q:	It is colder than yesterday
	r:	We go swimming
	s:	We take a canoe trip
	t:	We will be home by sunset

- $\neg p \land q$ — It is not sunny this afternoon and it is colder than yesterday
- $r \to p$ — We will go swimming only if it is sunny
- $\neg r \to s$ — If we do not go swimming, then we will take a canoe trip
- $s \to t$ — If we take a canoe trip, then we will be home by sunset

---

- t — We will be home by sunset

Using Rules of Inference

Hypothesis:

$\neg p \land q$

$r \to p$

$\neg r \to s$

$s \to t$

Conclusion:

t

Step		Reason
1.	$\neg p \land q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Premise
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \to s$	Premise
6.	s	Modus ponens using (4) and (5)
7.	$s \to t$	Premise
8.	t	Modus ponens using (6) and (7)

Therefore, the propositions can lead to the conclusion
We will be home by sunset

Using Rules of Inference

- Or, another presentation method:

Hypothesis:

$\neg p \wedge q$
$r \rightarrow p$
$\neg r \rightarrow s$
$s \rightarrow t$

Conclusion:

t

$(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$

$\Rightarrow \neg p \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ By Simplification

$\Rightarrow \neg r \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ By Modus Tollens

$\Rightarrow s \wedge (s \rightarrow t)$ By Modus Ponens

$\Rightarrow t$ By Modus Ponens

☺ Small Exercise ☺

- **Given:**
  - If you send me an e-mail message, then I will finish writing the program
  - If you do not send me an e-mail message, then I will go to sleep early
  - If I go to sleep early, then I will wake up feeling refreshed
- Can these propositions lead to the **conclusion** "If I do not finish writing the program, then I will wake up feeling refreshed."

Let p: you send me an e-mail message
q: I will finish writing the program
r: I will go to sleep early
s: I will wake up feeling refreshed

$p \rightarrow q$ — If you send me an e-mail message, then I will finish writing the program

$\neg p \rightarrow r$ — If you do not send me an e-mail message, then I will go to sleep early

$r \rightarrow s$ — If I go to sleep early, then I will wake up feeling refreshed

---

$\neg q \rightarrow s$ — If I do not finish writing the program, then I will wake up feeling refreshed

☺ Small Exercise ☺

Hypothesis:

$p \rightarrow q$

$\neg p \rightarrow r$

$r \rightarrow s$

Conclusion:

$\neg q \rightarrow s$

Step		Reason
1.	$p \rightarrow q$	Premise
2.	$\neg q \rightarrow \neg p$	Contrapositive of (1)
3.	$\neg p \rightarrow r$	Premise
4.	$\neg q \rightarrow r$	Hypothetical Syllogism using (2) and (3)
5.	$r \rightarrow s$	Premise
6.	$\neg q \rightarrow s$	Hypothetical Syllogism using (4) and (5)

Therefore, the propositions can lead to the conclusion
If I do not finish writing the program,
then I will wake up feeling refreshed

☺ Small Exercise ☺

- Or, another presentation method:

Hypothesis:

$p \to q$

$\neg p \to r$

$r \to s$

Conclusion:

$\neg q \to s$

$(p \to q) \land (\neg p \to r) \land (r \to s)$

$\Leftrightarrow (\neg q \to \neg p) \land (\neg p \to r) \land (r \to s)$ Contrapositive

$\Rightarrow (\neg q \to r) \land (r \to s)$ By Hypothetical Syllogism

$\Rightarrow (\neg q \to s)$ By Hypothetical Syllogism

Using Rules of Inference

Fallacies

- **Are the following arguments correct?**
  - **Example 1 (Fallacy of affirming the conclusion)**

    Hypothesis
    - If you success, you work hard — $p \to q$
    - You work hard — q

    Conclusion
    - You success — $\therefore p$ ✗

  - **Example 2 (Fallacy of denying the hypothesis)**

    Hypothesis
    - If you success, you work hard — $p \to q$
    - You do not success — $\neg p$

    Conclusion
    - You do not work hard — $\therefore \neg q$ ✗

Rules of Inference for Quantifiers

- **Universal Instantiation**

$$\frac{\forall x\, P(x)}{\therefore P(a)}$$

where a is a particular member of the domain

- **Existential Instantiation**

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$$

- **Universal Generalization**

$$\frac{P(b) \text{ for an arbitrary b}}{\therefore \forall x\, P(x)}$$

Be noted that b that we select must be an arbitrary, and not a specific

- **Existential Generalization**

$$\frac{P(d) \text{ for some element } d}{\therefore \exists x\, P(x)}$$

Rules of Inference for Quantifiers

- Example 1
  - **Given**
    - Everyone in this discrete mathematics class has taken a course in computer science
    - Marla is a student in this class
  - These premises imply the **conclusion** "Marla has taken a course in computer science"

Let	DC(x):	x studies in discrete mathematics
	CS(x):	x studies in computer science
	Domain of x:	student

$\forall x\ (DC(x) \rightarrow CS(x))$

- Everyone in this discrete mathematics class has taken a course in computer science

DC(Marla)

- Marla is a student in this class

CS(Marla)

- Marla has taken a course in computer science

Rules of Inference for Quantifiers

Premise:

$\forall x\ (DC(x) \rightarrow CS(x))$
DC(Marla)

Conclusion:

CS(Marla)

Step		Reason
1.	$\forall x\ (DC(x) \rightarrow CS(x))$	Premise
2.	$DC(Marla) \rightarrow CS(Marla)$	Universal Instantiation from (1)
3.	DC(Marla)	Premise
4.	CS(Marla)	Modus ponens using (2) and (3)

Therefore, the propositions can lead to the conclusion Marla has taken a course in computer science

Using Rules of Inference for Quantifiers

- Or, another presentation method:

Premise:

$\forall x\ (DC(x) \rightarrow CS(x))$

$DC(Marla)$

Conclusion:

$CS(Marla)$

$\forall x\ (DC(x) \rightarrow CS(x)) \wedge DC(Marla)$

By Universal Instantiation

$\Rightarrow (DC(Marla) \rightarrow CS(Marla)) \wedge DC(Marla)$

$\Rightarrow CS(Marla)$ By Modus ponens

☺ Small Exercise ☺

- **Given**
  - A student in this class has not read the book
  - Everyone in this class passed the first exam
- These premises imply the **conclusion** "Someone who passed the first exam has not read the book"

Let		
	C(x):	x in this class
	RB(x):	x reads the book
	PE(x):	x passes the first exam
	Domain of x:	any person

$\exists x\ (C(x) \land \neg RB(x))$

- A student in this class has not read the book

$\forall x\ (C(x) \rightarrow PE(x))$

- Everyone in this class passed the first exam

$\exists x\ (PE(x) \land \neg RB(x))$

- Someone who passed the first exam has not read the book

We cannot define the domain as student in this class since the conclusion means anyone

☺ Small Exercise ☺

Premise:

$\exists x\ (C(x) \land \neg RB(x))$

$\forall x\ (C(x) \rightarrow PE(x))$

Conclusion:

$\exists x\ (PE(x) \land \neg RB(x))$

Step	Reason
$\exists x\ (C(x) \land \neg RB(x))$	Premise
$C(a) \land \neg RB(a)$	Existential Instantiation from (1)
$C(a)$	Simplification from (2)
$\forall x\ (C(x) \rightarrow PE(x))$	Premise
$C(a) \rightarrow PE(a)$	Universal Instantiation from (4)
$PE(a)$	Modus ponens from (3) and (5)
$\neg RB(a)$	Simplification from (2)
$PE(a) \land \neg RB(a)$	Conjunction from (6) and (7)
$\exists x\ (PE(x) \land \neg RB(x))$	Existential Generalization from (8)

Therefore, the propositions can lead to the conclusion
Someone who passed the first exam has not read the book

☺ Small Exercise ☺

- Or, another presentation method:

$(\exists x\,(C(x) \wedge \neg RB(x))) \wedge (\forall x\,(C(x) \to PE(x)))$

$\Rightarrow C(a) \wedge \neg RB(a) \wedge (\forall x\,(C(x) \to PE(x)))$ By Existential Instantiation

$\Rightarrow C(a) \wedge \neg RB(a) \wedge (C(a) \to PE(a))$ By Universal Instantiation

$\Rightarrow PE(a) \wedge \neg RB(a)$ By Modus ponens

$\Rightarrow \exists x\,(PE(x) \wedge \neg RB(x))$ By Existential Generalization

Premise:
$\exists x\,(C(x) \wedge \neg RB(x))$
$\forall x\,(C(x) \to PE(x))$

Conclusion:
$\exists x\,(PE(x) \wedge \neg RB(x))$

Combining Rules of Inference

- The rules of inference of Propositions and Quantified Statements can be combined
  - **Universal Modus Ponens**

$\forall x\,(P(x) \to Q(x))$
$P(a)$, where a is a particular element in the domain
$\therefore\ Q(a)$

$(\forall x\,(P(x) \to Q(x))) \wedge (P(a))$
By **Universal** Instantiation
$\Rightarrow (P(a) \to Q(a)) \wedge (P(a))$
$\Rightarrow Q(a)$ By **Modus Ponens**

  - **Universal Modus Ponens**

$\forall x\,(P(x) \to Q(x))$
$\neg Q(a)$, where a is a particular element in the domain
$\therefore\ \neg P(a)$

$(\forall x\,(P(x) \to Q(x))) \wedge (\neg Q(a))$
By **Universal** Instantiation
$\Rightarrow (P(a) \to Q(a)) \wedge (\neg Q(a))$
$\Rightarrow \neg P(a)$ By **Modus Tollens**

Combining Rules of Inference

- Example:
  - **Given**
    - For all positive integers n,
      if n is greater than 4, then n^2 is less than 2^n

    is **true**.
  - **Show** that $100^2 < 2^{100}$

Combining Rules of Inference

- Example:

  For all positive integers n,
  if n is greater than 4, then n^2 is less than 2^n

P(n): $n > 4$
Q(n): $n^2 < 2^n$

$\forall n\,(P(n) \rightarrow Q(n))$

$P(100)$ (since $100 > 4$)

$\therefore$ Q(100) ($100^2 < 2^{100}$) **By Universal Modus Ponens**

Summary

- What we have learnt in previous lectures?
  - Proposition
  - Operator
  - Predicates
  - Quantifier
  - Truth Table
  - Rules of Equivalence
  - Rules of Inference

Show if an argument is valid

- This is called the **formal proof**
  - very clear and precise
  - extremely long and hard to follow

Informal Proofs

- **Informal proofs** can often explain to humans why theorems are true
  - Proof of mathematical theorems
  - Applications to computer science
- Move from formal proofs toward more **informal proofs**

Informal Proofs

- In practice, the proofs of theorems designed for human consumption are almost always **informal proofs**
  - More than one rule of inference may be used in each step
  - Steps may be skipped
  - The axioms being assumed
    - e.g. even number can be written as 2k, where k is integer
  - The rules of inference used are not explicitly stated

Proof for Theorems

- **Types of Theorem**
  - Implication $(P(x) \rightarrow Q(x))$
  - Equivalence $(P(x) \leftrightarrow Q(x))$
  - Statement $(P(x))$
- **Type of proof**
  - Universal Quantification (For all...)
  - Existential Quantification (For some...)
  - Uniqueness Quantification (Only one...)

Universal Quantification

Proof for Theorems: Methods

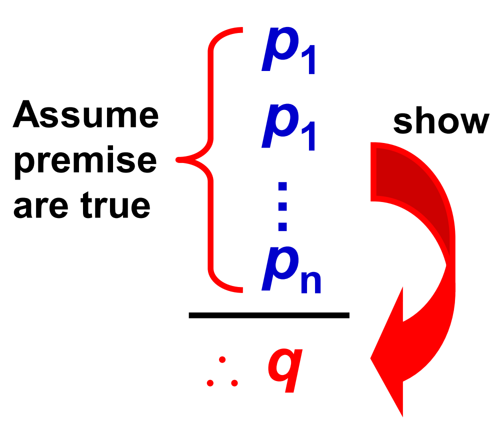
- **Implication** $(P(x) \rightarrow Q(x))$
  - **Direct Proof**
    Assume P(x) is true, show Q(x) is true
  - **Indirect Proof: Proof by Contraposition**
    Assume $\neg Q(x)$ is true and show $\neg P(x)$ is true
- **Equivalence** $(P(x) \leftrightarrow Q(x))$
  - As $P(x) \leftrightarrow Q(x) \equiv (P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x))$
- **Statement** $(P(x))$
  - **Indirect Proof: Proof by Contradiction**

Universal Quantification: Proof of Theorems: Implication

Direct Proof

- **Direct proofs** lead from the hypothesis of a theorem to the conclusion

1. Assume the premises are true
2. Show the conclusion is true



Direct Proof: Example 1

- Prove "If n is an odd integer, then n^2 is odd"
- Given,
  - The integer n is **even** if there exists an integer k such that n = 2k
  - The integer n is **odd** if there exists an integer k such that n = 2k+1

Show
If n is an odd integer, then n^2 is odd

1. Assume the hypothesis is true
   "n is odd" is true
   - By definition, n = 2k + 1, where k is a integer
2. Show the conclusion is correct
   n^2 is odd

   $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
   - By definition, as $(2k^2 + 2k)$ is an integer we can conclude that n^2 is an odd integer

- Therefore, "if n is an odd integer, then n^2 is an odd integer" has been proved

Direct Proof: Example 2

- Prove "If m and n are both perfect squares, then nm is also a perfect square"
- Given
  - An integer a is a **perfect square** if there is an integer b such that $a = b^2$

Show
If m and n are both perfect squares, then nm is also a perfect square

1. Assume m and n are both perfect squares
   - By definition, $m = a^2$ and $n = b^2$, where a and b are integers
2. Show that mn is a perfect square
   - $mn = a^2b^2 = (ab)^2$, where ab is an integer
   - By the definition, we can conclude that mn is a perfect square

- Therefore, "An integer a is a perfect square if there is an integer b such that $a = b^2$" has been proved

Direct Proof: Example 3

- Prove "if n is an integer and 3n + 2 is odd, then n is odd"
- Assume 3n + 2 is an odd integer
  - 3n + 2 = 2k + 1 for some integer k
- Show that n is odd

$$3n + 2 = 2k + 1$$
$$3n = 2k - 1$$
$$n = \frac{2k - 1}{3}$$

Indirect Proof

- Sometimes, direct proofs may reach dead ends
- **Indirect proof** may help
  - Prove theorems not directly
  - Do not start with the hypothesis and end with the conclusion

Proof by Contraposition

- Recall, contrapositive:

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

- $p \rightarrow q$ can be proved by showing $\neg q \rightarrow \neg p$ is true
  1. Assume the conclusion is not true
  2. Show either one premise is not true

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$$

$$\equiv \neg q \rightarrow \neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$$

$$\equiv \neg q \rightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$$

Proof by Contraposition: Example 1

- Prove "if n is an integer and 3n + 2 is odd, then n is odd"

$\neg q \rightarrow \neg p$

1. Assume the conclusion is false
   n is not odd
   - n = 2k, where k is an integer
2. Show that the premises are not correct
   3n + 2 is not odd
   - 3 (2k) + 2 = 6k + 2 = 2(3k + 2)

- As if n is not odd, 3n + 2 is not odd
  Therefore, if n is an integer and 3n + 2 is odd, then n is odd

Proof by Contraposition: Example 2

- Prove "if n = ab, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ "

1. Assume $a > \sqrt{n}$ and $b > \sqrt{n}$ is true
2. Show $n \neq ab$
   - $ab > (\sqrt{n})^2 = n$
   - Therefore, $ab \neq n$

- Therefore, if n = ab, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

☺ Small Exercise ☺

- Prove that "the sum of two rational numbers is rational"
- Given
  - The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that $r = p / q$
  - A real number that is not rational is called **irrational**

☺ Small Exercise ☺

- **Direct Proof**
  - Suppose that r and s are rational numbers
    - r = p / q, s = t / u, where $q \neq 0$ and $u \neq 0$
  - Show that r+s is rational number

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}$$

  - As $q \neq 0$ and $u \neq 0$, $qu \neq 0$
  - Therefore, r + s is rational
  - Therefore, direct proof succeeded

☺ Small Exercise ☺

- Prove "if n is an integer and n^2 is odd, then n is odd"

- **Direct proof**
  - Suppose that n is an integer and n^2 is odd
    - Exists an integer k such that $n^2 = 2k + 1$
  - Show n is odd
    - Show ($n = \pm\sqrt{2k + 1}$) is odd
    - May not be useful

☺ Small Exercise ☺

- **Proof by contraposition**
  - Suppose n is not odd
    - n = 2k, where k is an integer
  - Show n^2 is not even
    - $n^2 = (2k)^2 = 4k^2$
    - n^2 is even
  - Therefore, proof by contraposition succeeded

Universal Quantification

Proof of Theorems: Equivalence

- Recall, $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- To prove equivalence, we can show $p \rightarrow q$ and $q \rightarrow p$ are both true

Equivalence: Example

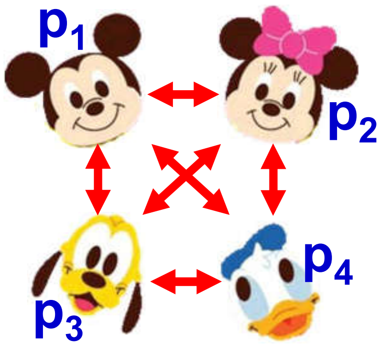
- Prove "If n is a positive integer, then n is odd **if and only if** n^2 is odd"

- Two steps
  1. If n is a positive integer, **if** n is odd, **then** n^2 is odd **(shown in slides 43)**
  2. If n is a positive integer, **if** n^2 is odd, **then** n is odd **(shown in slides 54)**

- Therefore, it is true

Proof of Theorems: Equivalence

- How to show p_1, p_2, p_3 and p_4 are equivalence?
  - $p_1 \leftrightarrow p_2$
  - $p_1 \leftrightarrow p_3$
  - $p_1 \leftrightarrow p_4$
  - $p_2 \leftrightarrow p_3$
  - $p_2 \leftrightarrow p_4$
  - $p_3 \leftrightarrow p_4$

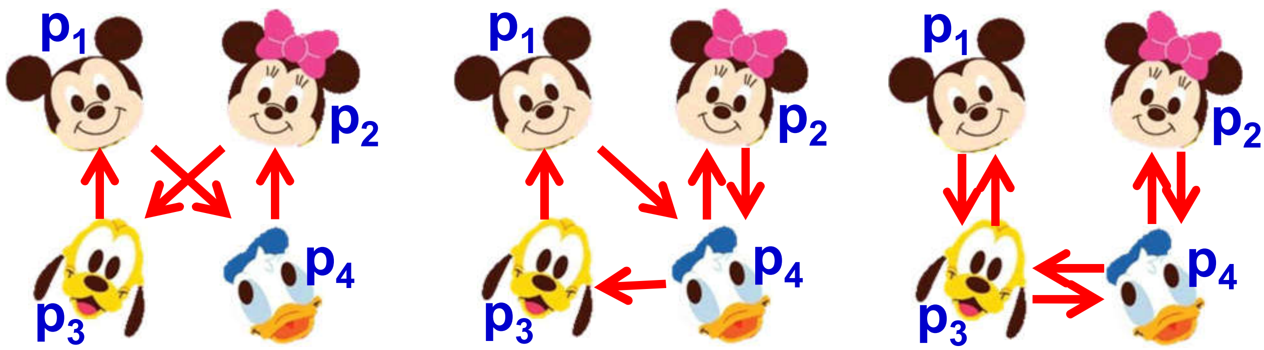


- **Not necessary**
  - E.g. if $p_1 \leftrightarrow p_2$ and $p_2 \leftrightarrow p_3$, then $p_1 \leftrightarrow p_3$

Universal Quantification
Proof of Theorems: Equivalence

$$p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow \ldots \leftrightarrow p_n$$

- When proving a group of statements are equivalent, any **chain of conditional statements** can established as long as it is possible to work through the chain to go from anyone of these statements to any other statement



Universal Quantification: Methods of Proving Theorems
Statement: Example

Trap??

Can you prove "You love me" ? **p**

If you love me, **$p \rightarrow q$**
you will buy me iphone5

How?

What does it mean if you…

1. Buy iphone **q** Prove nothing
2. Do not buy iphone

$\neg q \rightarrow \neg p$

Statement: Example (Correct)

Can you prove "You love me" ? **p**

If you do not love me,
you will not buy me iphone5

How?

$\neg p \rightarrow \neg q$

What does it mean if you…

1. Buy iphone **$q \rightarrow p$**
2. Do not buy iphone

$\neg q$ Prove nothing

Proof by Contradiction

- By using **Proof by Contradiction**,
  If you want to show p is true, you need:
  - $\neg p \rightarrow q$ is true
  - q is false

$\neg$P	Q	$\neg$P $\rightarrow$ Q
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

- Recall, Proof by Contradiction of $p \rightarrow q$ is
  $\neg p \rightarrow q$

Proof by Contradiction

- Procedures of Proof by Contradiction to prove p is correct :
  1. Understand the meaning of $\neg p$
  2. Find out what $\neg p$ implies ($\neg p \to q$ is true)
  3. Show that q is not correct

Proof by Contradiction: Example 1

- Prove $\sqrt{2}$ is irrational

Not "if… then…" format
Only one statement

1. **Understand the meaning of $\neg p$**

   $\sqrt{2}$ is rational

2. **Find out what $\neg p$ implies**

   q

   If $\sqrt{2}$ is rational, there exist integers p and q with $\sqrt{2}$ = p / q, where p and q have no common factors

   - So that the fraction p / q is in lowest terms

3. **Show that q is not correct**

   Show "there exist integers p and q with $\sqrt{2}$ = p / q" is not true

Show “there exist integers p and q with $\sqrt{2} = p / q$” is not true

$$\sqrt{2} = p / q \quad , \text{ where } q \neq 0$$

$$2q^2 = p^2$$

- p^2 is an even number
- If p^2 is even, so p = 2a, and a is an integer

$$2q^2 = 4a^2$$

$$q^2 = 2a^2$$

- q is also even
- As p and q are even, they have a common factor 2, which leads the **contradiction**
- Therefore, “ $\sqrt{2}$ is irrational” is true

Proof by Contradiction: Example 2

- Show that at least four of any 22 days must fall on the same day of the week.

July

W	S	M	T	W	T	F	S
19						1	2
20	3	4	5	6	7	8	9
21	10	11	12	13	14	15	16
22	17	18	19	20	21	22	23
23	24	25	26	27	28	29	30
24	31						

Let p: "At least four of 22 chosen days fall on the same day of the week."

1. **Understand the meaning of ¬p**

   At most three of 22 chosen days fall on the same day of the week

2. **Find out what ¬p implies**

   As at most three day fall on the same week day, therefore a week should have at least 22 / 3 days

3. **Show that q is not correct**

   A week only has 7 days, therefore, q is not correct

Therefore, p is correct

Proof by Contradiction

- Proof by Contradiction can also be used to show $P(x) \rightarrow Q(x)$ (implication)
- Let $S(x) : P(x) \rightarrow Q(x)$ and prove $S(x)$ is correct
  - $S(x) : P(x) \rightarrow Q(x)$
  - $\neg S(x) \rightarrow (P(x) \wedge \neg Q(x))$ is true
  - $P(x) \wedge \neg Q(x)$ is false

$$\begin{aligned} &\neg S(x) \\ &= \neg(P(x) \rightarrow Q(x)) \\ &= \neg(\neg P(x) \vee Q(x)) \\ &= P(x) \wedge \neg Q(x) \end{aligned}$$

Proof by Contradiction: Example 3

- Show "If 3n + 2 is odd, then n is odd“
  Be noted that proof by contraposition can be used (shown in slide 50)

- Let $P(n): Q(3n+2) \rightarrow Q(n)$,
  where Q(n) : “n is odd”

- $\neg P(n)$ implies:

$$\neg P(n) \equiv \neg(Q(3n+2) \rightarrow Q(n))$$
$$\equiv \neg(\neg Q(3n+2) \vee Q(n))$$
$$\equiv Q(3n+2) \wedge \neg Q(n)$$

Universal Quantification: Methods of Proving Theorems: Statement

Proof by Contradiction: Example 3

- $\neg P(n)$ implies “$Q(3n+2) \wedge \neg Q(n)$”
  - $\neg Q(n)$ imply…
    - n is even, n = 2k, where k is integer
    - $3n+2 = 3(2k)+2 = 2(3k+1)$
    - Therefore, 3n+2 is even ($\neg Q(3n+2)$)
    - $Q(3n+2) \wedge \neg Q(3n+2)$ is false
    - Therefore, $\neg P(n)$ must be false
  - Therefore,
    - $Q(3n+2) \rightarrow Q(n)$ is true

Exhaustive Proof and Proof by Cases

- Sometimes, a theorem cannot be proved easily using a single argument that holds for all possible cases
- Rather than considering ($p \rightarrow q$) directly, we can consider different cases separately
- This argument is named **Proof by Cases**:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

$$\equiv [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

- E.g. $x^2 \geq 0$, we can $x < 0$, $x = 0$ and $x > 0$

Exhaustive Proof

- **Exhaustive Proofs**
  - Prove all the possibilities
  - Example
    - Prove that $(n + 1)^3 > 3^n$ if n is a positive integer with $n \leq 4$
    - Prove all the possibilities: $n = 1, 2, 3$ and 4

Exhaustive Proof: Example 1

- Prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$

When n = 1
LHS: $(n + 1)^3 =$ **8**
RHS: $3^n =$ **3**
LHS > RHS

When n = 2
LHS: $(n + 1)^3 =$ **27**
RHS: $3^n =$ **9**
LHS > RHS

When n = 3
LHS: $(n + 1)^3 =$ **64**
RHS: $3^n =$ **27**
LHS > RHS

When n = 4
LHS: $(n + 1)^3 =$ **125**
RHS: $3^n =$ **81**
LHS > RHS

- Therefore, $(n + 1)^3 > 3^n$ is valid

Exhaustive Proof: Example 2

- Given
  - An integer is a **perfect power** if it equals n^a , where a is an integer greater than 1
- Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9
  - By exhaustive proof, list all the perfect powers not exceeding 100

	n=1	2	3	4	5	6	7	8	9	10
a=2	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**
3	**1**	**8**	**27**	**64**						
4	**1**	**32**	**81**							
5	**1**	**64**								
>5	**1**									

  - Therefore, only 8 and 9 are consecutive

Proof by Cases

- **Drawback** of Exhaustive Proofs is to check only a relatively small number of instances of a statement
- **Proof by Cases**
  - Prove all situations
  - Example
    - Prove that if n is an integer, then $n^2 > n$
    - Prove all the situations: n is positive, equal and negative

Proof by Cases: Example 1

- Prove that if n is an integer, then $n^2 \geq n$

When $n \geq 1$

$n^2 = n \times n \geq n \times 1 = n$, therefore $n^2 \geq n$

When $n = 0$

$n^2 = n = 0$, therefore, $n^2 = n$

When $n \leq -1$

$n^2 > 0$ and $n < 0$, therefore $n^2 \geq n$

- Therefore, this theorem is valid

Proof by Cases: Example 2

- Use a proof by cases to show that **| x y | = |x| |y|**, where x and y are real numbers

  (Recall **|a| = a**, when $a \geq 0$; **|a| = -a** when $a < 0$)

When x ≥ 0 and y ≥ 0

| x y | = x y = |x| |y|

When x ≥ 0 and y < 0

| x y | = - x y = (x) (-y)= |x| |y|

When x < 0 and y ≥ 0

| x y | = - x y = (-x) (y)= |x| |y|

When x < 0 and y < 0

| x y | = x y = (-x) (-y)= |x| |y|

- Therefore, this theorem is valid

Existence Proofs

- We will focus on the theorems which are assertions that objects of a particular type exist (∃)
  - A theorem of this type is a proposition of the form $\exists x\ P(x)$, where P is a predicate
  - The proof of this proposition is **Existence Proof**
    - By finding an element a such that P(a) is true

Existence Proofs

- Example:
  - Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways
  - After considerable computation (such as a computer search), we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

  - An example is given, the proof is done

Uniqueness Proof

- The theorems which assert the existence of a unique element with a particular property will be discussed
- The **two parts** of a uniqueness proof are:
  - **Existence** (An element with the property exists)
    - Show that an element x with the property exists
  - **Uniqueness** (No other element has this property)
    - Show that if $y \neq x$, y does not have the property.
- Equivalently, we can show that if x and y both have the desired property, then $x = y$

$$\underbrace{\exists x\, (P(x)}_{\textbf{Existence}} \wedge \underbrace{\forall y (\, P(y) \rightarrow (y = x)\,)}_{\textbf{Uniqueness}}\,)$$

Uniqueness Proof: Example

- Show that if a and b are real numbers and a ≠ 0, then there is a unique real number r such that ar + b = 0

- **Existence Part**
  - The real number **t = -b / a** is a solution of ar + b = 0 because a(-b/a) + b = -b + b = 0
  - Consequently, a real number t exists for which at + b = 0
- **Uniqueness Part**
  - Suppose that s is a real number such that as + b = 0

$$at + b = as + b \quad\quad t \text{ is } -b / a$$
$$at = as \quad\quad a \text{ is nonzero}$$
$$t = s$$

  - This means that if s ≠ t, then as + b ≠ 0

Tips

- **DO NOT over simplify the proof**
  - “Obviously" or "clearly” in proofs indicate that steps have been omitted that the author expects the reader to be able to fill in
  - Unfortunately, this assumption is often not warranted
  - We will assiduously try to avoid using these words and try not to omit too many steps
- However, if we included all steps in proofs, our proofs would often be **too long**