Chapter 1: Logic and Proof

1.3
Predicates and Quantifiers

1.4
Nested Quantifiers

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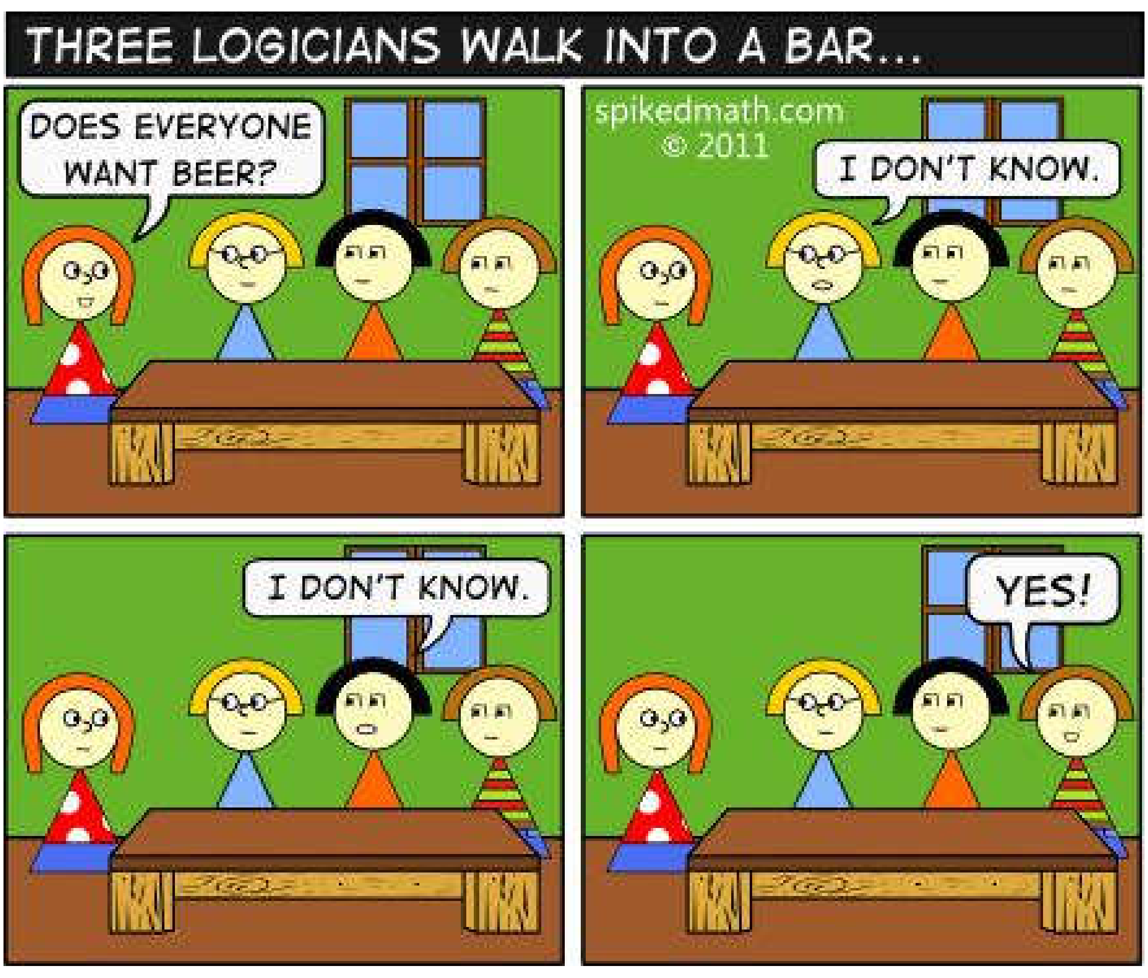


Agenda

- Ch1.3 Predicates and Quantifiers
  - Predicates
  - Quantifiers
  - Quantifiers with Restricted Domains
  - Precedence of Quantifiers
  - Logical Equivalences Involving Quantifiers
  - Translation
- Ch1.4 Nested Quantifiers
  - Nested Quantifiers

Limitation of Propositional Logic

- Limitation 1:

p : John is a SCUT student

q : Peter is a SCUT student

r : Mary is a SCUT student

- Try to represent them using propositional variable
  - However, these propositions are very similar
  - A more powerful type of logic named **Predicate Logic** will be introduced

Predicates

- **Predicate logic** is an extension of propositional logic that permits concisely reasoning about whole classes of entities

John Peter Mary

????

- **Propositional Logic** treats simple propositions as atomic entities
- **Predicate Logic** distinguishes the subject of a sentence from its predicate

Predicates

- **Predicate** is a function of proposition
- Example:

Convention:
- lowercase variables denote objects
- UPPERCASE variables denote predicates

Propositional Function / Predicate — $P(x)$

Variable — x

Predicate — is greater than 3

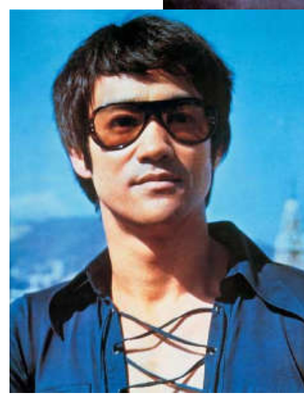
$P(x)$: x is greater than 3

Refer to

- The truth value of proposition function can only be determined when the values of variables are known

Predicates

- Example:
  - *P*(*x*) : "*x* > 3"
  - What is ***P*(4)**? ✓
  - What is ***P*(2)**? ✗



  - *P*(*x*) : "*x* is a singer"
  - ***P*(Michael Jackson)**? ✓
  - ***P*(Bruce Lee)**? ✗

Predicates

- Propositional function can have more than one variables
- Example:
  - P(x, y): x + y = 7
    - P(2, 5) ✓
  - Q(x, y, z): x = y + z
    - Q(5, 2, 8) ✗

Predicates

- **General case**
  - A statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n)$$

  - A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the **n-tuple** $(x_1, x_2, \ldots, x_n)$
  - P is also called a ***n*-place predicate** or a ***n*-ary predicate**

Limitation of Propositional Logic

- Limitation 2:
  - Given

    P: "Every student in SCUT is clever"

    Q: "Peter is SCUT student"
  - What can we conclude?

    "Peter is clever"
  - Given

    P: "Peter cannot pass this Discrete Maths subject"

    Q: "Peter is a SCUT student"
  - What can we conclude?

    "At least one student in SCUT cannot pass this Discrete Maths subject"
  - **No rules of propositional logic** can conclude the truth of this statement

Limitation of Propositional Logic

- **Propositional Logic** does not adequately express the following meanings
  - Every, all, some, partial, at least one, one, etc

- A more powerful tool, **Quantifiers**, will be introduced

Quantifiers

- **Quantification** expresses the extent to which a predicate is true over a range of elements
- For example
  - Using Propositional Logic
    - p: Peter has iPhone
    - q: Paul has iPhone
    - r: Mary has iPhone
  - Using Predicate
    - P(x) : x has iPhone
    - P(Peter)
    - P(Paul)
    - P(Mary)

Assume our class only contains three students

- **Using Quantifier**
  - P(x) : x has iPhone
  - For all x, P(x) is true
  - Domain consists of all student in this class

Quantifiers

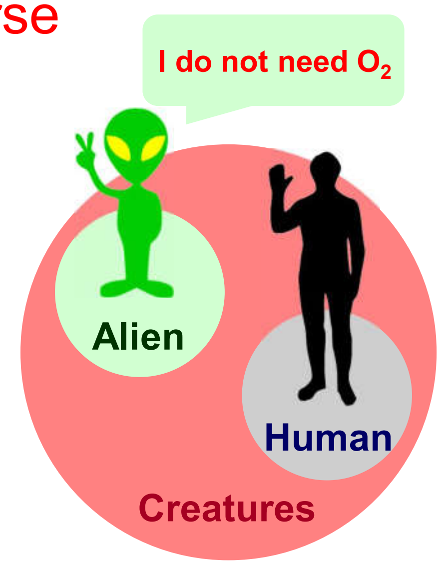
- Four aspects should be mentioned in Quantification
  1. Quantifier
     (e.g. all, some…)
  2. Variable
  3. Predicate
  4. Domain

P(x) : x has iPhone
For all x, P(x) is true
Domain consists of all student in this class

- The area of logic that deals with predicates and quantifiers is called the **Predicate Calculus**

Quantifiers

- **Universes of Discourse (U.D.s)**
  - Also called the **domain of discourse**
  - Refers to the collection of objects being discussed in a specific discourse
  - Example:
    - *P*(*x*) : "*x* breaths oxygen"
    - Domain consists of **humans**
      P(*x*) is true for all x? ✓
    - Domain consists of **creatures**
      P(*x*) is true for all x? ✗



Quantifiers

- Three types of quantification will be focused:
  - **Universal Quantification**
    - i.e. all, none
  - **Existential Quantification**
    - i.e. some, few, many
  - **Unique Quantification**
    - i.e. exactly one
    - Can be expressed by using Universal Quantification and Existential Quantification

Universal Quantifiers (ALL)

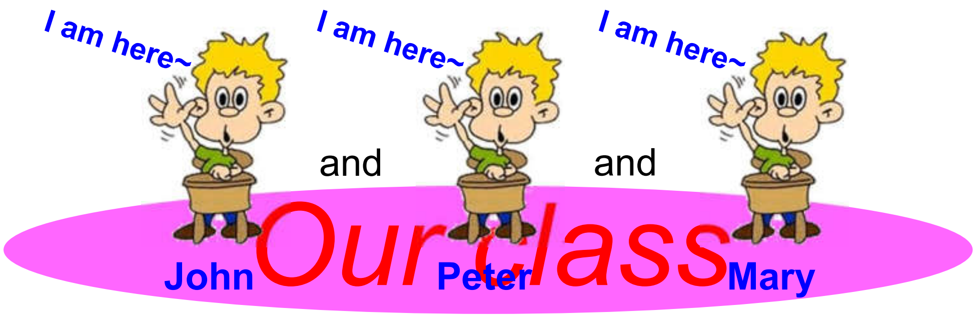
- Definition
  Universal quantification of $P(x)$ is the statement
  "$P(x)$ is true for all values of x in the domain"
- Notation: $\forall x\ P(x)$
  - $\forall$LL, reversed "A"
  - Read as
    - "for all x $P(x)$"
    - "for every x $P(x)$"
- Truth value
  - **True** when $P(x)$ is true for all x
  - **False** otherwise
    - An **element** for which $P(x)$ is **false** is called a counterexample

Universal Quantifiers

- When all of the elements in the universe of discourse can be listed one by one (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\forall x\, P(x) \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

- For example
  - **Our class** has three students: **John, Peter and Mary**
  - **Every student in our class has attended the class**



Existential Quantifiers (SOME)

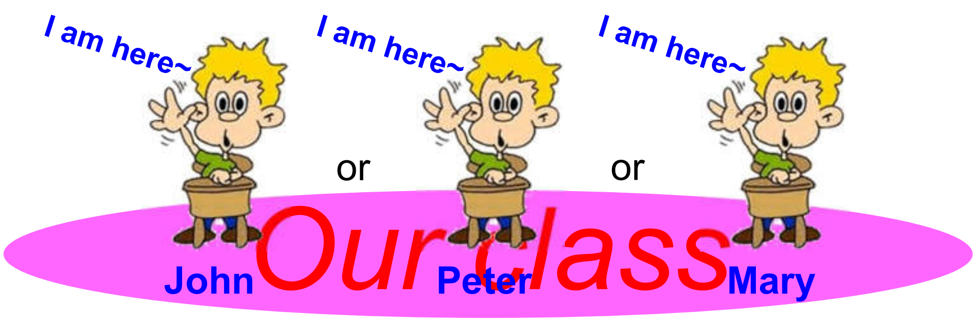
- Definition
  Existential quantification of P(x) is the proposition
  "There exists an element x in the domain such that $P(x)$ is true"
- Notation: $\exists x\, P(x)$
  - ∃XIST, reversed "E"
  - Read as
    - "There is an x such that $P(x)$"
    - "There is at least one x such that $P(x)$"
    - "For some x $P(x)$"
- Truth value
  - **False** when $P(x)$ is false for all x
  - **True** otherwise

Existential Quantifiers

- When all of the elements in the universe of discourse can be listed one by one (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

- For example
  - **Our class** has three students: **John, Peter and Mary**
  - **Any student in our class has attended the class**



Quantifiers

- Examples:
  - ***P*(*x*): *x*+1>*x*,** U.D.s: the set of real number
    - $\forall x\, P(x)$? **True**
    - $\exists x\, P(x)$? **True**

    $P(x)$ is always true
  - ***Q*(*x*): *x*<2,** U.D.s: the set of real number
    - $\forall x\, Q(x)$? **False** — $Q(y)$ is false when $y \geq 3$ (counterexamples)
    - $\exists x\, Q(x)$? **True** — $Q(y)$ is true when $y < 2$
  - ***S*(*x*): 2*x*<*x*,** U.D.s: the set of real positive number
    - $\forall x\, S(x)$? **False**
    - $\exists x\, S(x)$? **False**

    $S(x)$ is always false

Universal Quantifiers

- Examples:
  - ***P*(*x*): x^2<10,**
    U.D.s. the positive integer not exceeding 4
    - $\forall x\, P(x)$?

      $$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \;\equiv \mathbf{F}$$

    - $\exists x\, P(x)$?

      $$\exists x\, P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \;\equiv \mathbf{T}$$

$P(1)$ ✓ $P(2)$ ✓ $P(3)$ ✓ $P(4)$ ✗

counterexample

Quantifiers

- How can we prove the followings:
  - Universal quantification is true
  - Universal quantification is false
  - Existential quantification is true
  - Existential quantification is false

Attend the class?

Finding one is ok (counterexample)

Need to consider ALL

Statement	When true?	When false?
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which P(x) is false.
$\exists x\, P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Precedence of Quantifiers

- Recall,

Precedence	Operator	
1	$\neg$	NOT
2	$\wedge$	AND
3	$\vee$ $\oplus$	OR XOR
4	$\rightarrow$	Imply
5	$\leftrightarrow$	Equivalent

- $\forall$ and $\exists$ have higher precedence than all logical operators from proposition calculus

- Example
  - **$\forall x\ P(x) \wedge Q(x)$**

    $(\forall x P(x)) \wedge Q(x)$ ✓ $\forall x\ (P(x) \wedge Q(x))$ ✗

☺ Small Exercise ☺

- How to interpret the following expression:
  - $\forall x\ (P(x) \wedge \exists z\ Q(x,z) \rightarrow \exists y\ R(x,y)) \vee Q(x,y)$
  - $\forall x\ (\ P(x) \wedge \underbrace{(\exists z\ Q(x,z))} \rightarrow \underbrace{(\exists y\ R(x,y))}\ ) \vee Q(x,y)$
  - $\forall x\ (\ \underbrace{(P(x) \wedge \underbrace{(\exists z\ Q(x,z))})} \rightarrow \underbrace{(\exists y\ R(x,y))}\ ) \vee Q(x,y)$

Bound and Free Variable

- **Free Variable**: No any restriction
- **Bound Variable**: Some restrictions (quantifier or condition)
- Example:
  - $P(x)$: "$x > 3$" **Free Variable** Not Proposition
  - $P(x)$: "$x > 3$" and $x = 4$ **Bound Variable** Proposition
  - $\forall x\, P(x, y)$ ***x*:Bound Variable** **y:Free Variable** Not Proposition
- All the variables that occur in a quantifier must be bounded to turn it into a proposition
  - i.e. the truth value can be determined
- Giving restrictions on a free variable is called ***blinding***

Scope

- The part of a logical expression to which a quantifier is applied is called the ***scope* of this quantifier**
- For example

$$\forall x\, \underbrace{(P(x) \wedge (\exists y\, \overbrace{Q(y)}^{\text{Scope of } \exists y}))}_{\text{Scope of } \forall x} \vee R(z)$$

☺ Small Exercise ☺

- $\forall x \, (\, (P(x) \wedge (\exists z \, Q(x,z))) \rightarrow (\exists y \, R(x,y)) \,) \vee Q(x,y)$
  - Scope of $\exists z$: $Q(x,z)$
  - Scope of $\exists y$: $R(x,y)$
  - Scope of $\forall x$: $P(x) \wedge \exists z \, Q(x,z) \rightarrow \exists y \, R(x,y)$
  - Free Variable: x, y in $Q(x,y)$
  - Bound Variable: x, y, z in the first component

☺ Small Exercise ☺

- $\forall x \, \exists x \, P(x)$ **Not a free variable**
  - Any problem?

    x is not a free variable in $\exists x \, P(x)$, therefore the $\forall x$ binding is not used
- $\forall x \, P(x) \wedge Q(x)$ **1st x is Bounded variable 2nd x is Free variable**
  - Is x a free variable?

    The variable x in $Q(x)$ is outside of the *scope* of the $\forall x$ quantifier, and is therefore free
- $(\forall x \, P(x)) \wedge (\exists x \, Q(x))$ **Different variables**
  - Are x the same?
  - This is legal, because there are 2 different x

Recall....

- $\exists x\ (x^2>1)$
  - Domain of x is real number ✓
  - Domain of x is between -1 and 1 ✗
- $\forall x\ (x^2 \geq 1)$
  - Domain of x is integer ✗
  - Domain of x is positive integer ✓

Recall, the Equivalences

- Two propositions P and Q are logically equivalent if $P \leftrightarrow Q$ is a tautology
- $P \leftrightarrow Q$ means $(P \rightarrow Q) \wedge (Q \rightarrow P)$
  - $(P \rightarrow Q)$: Given P, Q is true
  - $(Q \rightarrow P)$: Given Q, P is true
- Therefore,
  if we want to show **$P \equiv Q$**,
  we can show $P \rightarrow Q$ and $Q \rightarrow P$

Universal Quantification

- $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$
  - $\forall x\, (P(x) \wedge Q(x)) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✓	✓	✓	✓
Q(x): x likes beer	✓	✓	✓	✓

  - $\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\, (P(x) \wedge Q(x))$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✓	✓	✓	✓
Q(x): x likes beer	✓	✓	✓	✓

Universal Quantification

- ~~$\forall x\, (P(x) \vee Q(x)) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$~~
  - ~~$\forall x\, (P(x) \vee Q(x)) \rightarrow \forall x\, P(x) \vee \forall x\, Q(x)$~~

	Peter	John	Mary	Jessica
P(x): x is lazy	✗	✓	✓	✓
Q(x): x likes beer	✓	✗	✗	✗

  - $\forall x\, P(x) \vee \forall x\, Q(x) \rightarrow \forall x\, (P(x) \vee Q(x))$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✗	✗	✗	✗
Q(x): x likes beer	✓	✓	✓	✓

Quantifiers: Logical Implication & Equivalence
Existential Quantification

- ~~$\exists x\,(P(x) \land Q(x)) \equiv \exists x\,P(x) \land \exists x\,Q(x)$~~
  - $\exists x\,(P(x) \land Q(x)) \rightarrow \exists x\,P(x) \land \exists x\,Q(x)$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✗	✓	✗	✓
Q(x): x like beer	✗	✓	✗	✗

  - ~~$\exists x\,P(x) \land \exists x\,Q(x) \rightarrow \exists x\,(P(x) \land Q(x))$~~

	Peter	John	Mary	Jessica
P(x): x is lazy	✓	✗	✗	✗
Q(x): x like beer	✗	✓	✗	✗

Quantifiers: Logical Implication & Equivalence
Existential Quantification

- $\exists x\,(P(x) \lor Q(x)) \equiv \exists x\,P(x) \lor \exists x\,Q(x)$
  - $\exists x\,(P(x) \lor Q(x)) \rightarrow \exists x\,P(x) \lor \exists x\,Q(x)$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✗	✗	✗	✗
Q(x): x like beer	✗	✓	✗	✗

  - $\exists x\,P(x) \lor \exists x\,Q(x) \rightarrow \exists x\,(P(x) \lor Q(x))$ ✓

	Peter	John	Mary	Jessica
P(x): x is lazy	✗	✗	✓	✗
Q(x): x like beer	✗	✗	✗	✗

Quantifiers

Logical Implication & Equivalence

- **For Universal Quantifiers,**
  - $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$
  - $\forall x\, P(x) \vee \forall x\, Q(x) \rightarrow \forall x\, (P(x) \vee Q(x))$
- **For Existential Quantifiers,**
  - $\exists x\, (P(x) \wedge Q(x)) \rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$
  - $\exists x\, (P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

Quantifiers: Logical Equivalence

- $\forall x\, (A \wedge P(x)) \equiv A \wedge \forall x\, P(x)$
- $\forall x\, (A \vee P(x)) \equiv A \vee \forall x\, P(x)$
- $\exists x\, (A \wedge P(x)) \equiv A \wedge \exists x P(x)$
- $\exists x\, (A \vee P(x)) \equiv A \vee \exists x P(x)$
- $\forall x\, P(x) \rightarrow A \equiv \exists x\, (P(x) \rightarrow A)$
- $A \rightarrow \forall x\, P(x) \equiv \forall x\, (A \rightarrow P(x))$
- $\exists x\, P(x) \rightarrow A \equiv \forall x\, (P(x) \rightarrow A)$
- $A \rightarrow \exists x\, P(x) \equiv \exists x\, (A \rightarrow P(x))$

* A does not consist of free variable x

$$
\begin{aligned}
&\forall x P(x) \rightarrow A \\
&\equiv \neg(\forall x P(x)) \vee A \\
&\equiv \exists x\, (\neg P(x)) \vee A \\
&\equiv \exists x\, (\neg P(x) \vee A) \\
&\equiv \exists x\, (P(x) \rightarrow A)
\end{aligned}
$$

$$
\begin{aligned}
&A \rightarrow \forall x P(x) \\
&\equiv \neg(A) \vee \forall x P(x) \\
&\equiv \forall x(\neg(A) \vee P(x)) \\
&\equiv \forall x(A \rightarrow P(x))
\end{aligned}
$$

Negating Quantifiers
Universal Quantification

- De Morgan’s Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists ? x\, \neg P(x)$$

Not all students are good There is a student is bad

P(x): x is a good student

Negating Quantifiers
Existential Quantification

- De Morgan’s Laws for Quantifiers

$$\neg\exists x\, P(x) \equiv \forall ? x\, \neg P(x)$$

There is not exist a good student All students are bad

P(x): x is a good student

☺ Small Exercise ☺

- What are the negation of the following statements?
  - $\forall x\ (x^2>x)$
    - $\neg\forall x(x^2>x) \equiv$
  - $\exists x\ (x^2=2)$
    - $\neg\exists x(x^2=2) \equiv$

☺ Small Exercise ☺

- Show that

$$\neg\forall x(P(x)\rightarrow Q(x)) \equiv \exists x(P(x)\wedge\neg Q(x))$$

$$\neg\forall x\ (P(x)\rightarrow Q(x))$$

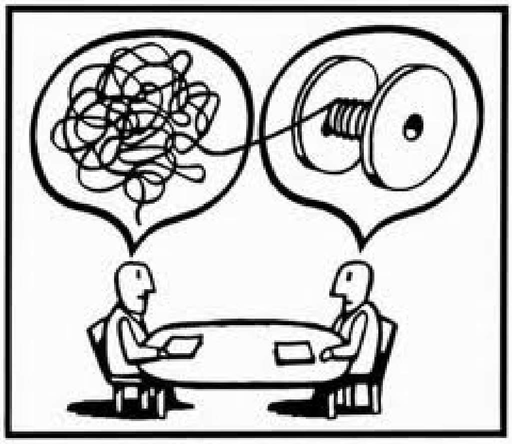
$$\equiv \neg\forall x\ (\neg P(x)\vee Q(x))$$

$$\equiv \exists x\ \neg(\neg P(x)\vee Q(x))$$

$$\equiv \exists x\ (P(x)\wedge\neg Q(x))$$

Translation Using Quantifiers

- Translating from **English** to **Logical Expressions with quantifiers**



Translation Using Quantifiers
Universal Quantification

- Using predicates and quantifiers, express the statement

Every | student | in this class | is lazy

∀ | ***x*** | Universe of Discourse | Predicate

  - Quantifier: **Universal Quantifier**
  - Variable: ***x***
  - Universe of discourse: **the students in the class**
  - Propositional Function: **P(x) : x is lazy**
  - Answer: $\forall x\, P(x)$

Universal Quant

The universal quantifier connects with a implication

- Another way to express the statement:

Every | student | in this class | is lazy
$\forall$ | x | Predicate (Q) | Predicate (P)

  - Quantifier: **Universal Quantifier**
  - Variable: x
  - Universe of discourse: **Any person**
  - Propositional Function: **P(x): x is lazy**
    Q(x): x is a student in this class

- Answer:

  $\forall x\ (Q(x) \rightarrow P(x))$ ✓

  For every person, if he/she is in this class, he/she is lazy

  $\forall x\ (Q(x) \wedge P(x))$ ✗

  For every person, he/she is in this class and lazy

Existential Quantification

- Using predicates and quantifiers, express the statement

Some | students | in this class | are lazy
$\exists$ | x | Universe of Discourse | Predicate

  - Quantifier: **Existential Quantifier**
  - Variable: x
  - Universe of discourse: **the students in the class**
  - Propositional Function: **P(x) : x is lazy**
  - Answer: $\exists x\ P(x)$

Existential Quan[tifier]

The existential quantifier connects with a conjunction

- Another way to express the statement:

  Some students in this class are lazy

  ∃ x Predicate (Q) Predicate (P)

  - Quantifier: **Existential Quantifier**
  - Variable: ***x***
  - Universe of discourse: **Any person**
  - Propositional Function: **P(x): x is lazy**
    Q(x): x is a student in this class

- Answer:

  $\exists x\,(Q(x) \rightarrow P(x))$ ✗

  Include the case which contains no person in this class

  For some persons, if he/she is in this class, he/she is lazy

  $\exists x\,(Q(x) \wedge P(x))$ ✓

  For some persons, he/she is in this class and lazy

☺ Small Exercise ☺

- Using predicates and quantifiers, set the domain as
  1. Staff in IBM company
  2. Any persons

  express the following statements:
  - Every staff in IBM company has visited Mexico
  - Some staff in IBM company has visited Canada or Mexico

☺ Small Exercise ☺

- Every staff in IBM company has visited Mexico
  - **Solution 1:**
    - Universal Quantifier
    - Variable: x
    - U.D.: Staffs in IBM company
    - Let $P(x)$: x has visited Mexico
    - $\forall x\, P(x)$
  - **Solution 2:**
    - Universal Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $Q(x)$: x is a staff in IBM company
    - Let $P(x)$: x has visited Mexico
    - $\forall x\, (Q(x) \rightarrow P(x))$

☺ Small Exercise ☺

- Some staff in IBM company has visited Canada or Mexico
  - **Solution 1:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Staffs in IBM company
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\, (P(x) \vee Q(x))$
  - **Solution 2:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a staff in IBM company
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\, (S(x) \wedge (P(x) \vee Q(x)))$

☺ Small Exercise ☺

- Some students in this class has visited Canada or Mexico
  - **Better Solution:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a student in this class
    - Let $P(x, loc)$: x has visited loc
    - $\exists x\, (S(x) \wedge (P(x, Canada) \vee P(x, Mexico)))$
  - **Solution 2:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a student in this class
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\, (S(x) \wedge (P(x) \vee Q(x)))$

Quantifiers with Restricted Domains

- An **abbreviated notation** is often used to restrict the domain of a quantifier
- Example
  - the square of any real number which greater than 10 is greater than 100
  - Using Domain
    $\forall x\, (x^2 > 100)$,
    U.D.s: the set of real number which is bigger than 10
  - Using Predicate
    $\forall x\, (x>10 \rightarrow x^2>0)$, U.D.s: the set of real number
  - Using Abbreviated Notation
    $\forall x{>}10\, (x^2>100)$, U.D.s: the set of real number

Quantifiers with Restricted Domains

- Example
  - Given that the domain in each case consists of the real number, what do the following statements mean?
    - $\forall x<0\ (x^2>0)$

      The square of negative real number is positive
    - $\forall y\neq0\ (y^3\neq0)$

      The cube of nonzero real number is nonzero
    - $\exists z>0\ (z^2=2)$

      There is a positive square root of 2

☺ Small Exercise ☺

- Using predicates and quantifiers, express the following statements:
  - Every mail message larger than one megabyte will be compressed
  - If a user is active, at least one network link will be available.

☺ Small Exercise ☺

- Every mail message larger than one megabyte will be compressed
- Solution:
  - Let S(***m***, ***y***) be "Mail message ***m*** is larger than ***y*** megabytes"
    - **Domain** of ***m*** :
    - **Domain** of ***y*** :
  - Let C(***m***) denote "Mail message ***m*** will be compressed"
  - $\forall m\ (S(m, 1) \rightarrow C(m))$

☺ Small Exercise ☺

- If a user is active, at least one network link will be available.
- Solution
  - Let A(***u***) be "User ***u*** is active"
    - **Domain** of ***u*** :
  - Let S(***n***, ***x***) be "Network link ***n*** is in state ***x***"
    - **Domain** of ***n*** :
    - **Domain** of ***x*** :
  - $\exists u\ A(u) \rightarrow \exists n\ S(n, \text{available})$

Nested Quantifiers

- Two quantifiers are **nested** if one is within the scope of the other
- How to interpret it?
  - If quantifiers are **same** type, the **order is not a matter**
    - $\exists x \exists y$ "x+y=0"
    - $\exists y \exists x$ "x+y=0"

    } Same meaning
  - If quantifiers are **different** types, read **from left to right**
    - $\forall x \exists y$ "x+y=0"
    - $\exists y \forall x$ "x+y=0"

    } Different meaning

Nested Quantifiers
Different Type

- If quantifiers are different types, read from left to right
- Example 1:
  - **P(x, y) = "x loves y"**
    $\forall x \exists y$ P(x, y) VS $\exists y \forall x$ P(x, y)

- **$\forall x \exists y$ "x loves y"**
  - For all x, there is at least one y, to make P(x,y) happens
  - For all persons, there is a person they love
  - ALL people loves some people

- **$\exists y \forall x$ "x loves y"**
  - At least one y, all x, to make P(x,y) happens
  - There is a person who is loved by all persons
  - Some people are loved by ALL people

Different Type

- Example 2:
  - **P(x, y) = "x+y=0"**
    $\forall x\ \exists y$ P(x, y) VS $\exists y\ \forall x$ P(x, y)

- **$\forall x\ \exists y$ (x+y=0)** ✓
  - For all x, there is at least one y, to make P(x,y) happens
  - Every real number has an additive inverse

- **$\exists y\ \forall x$ (x+y=0)** ✗
  - At least one y, all x, to make P(x,y) happens
  - There is a real number which all real number are its inverse addition

Same Type

- If quantifiers are the same type, the order is not a matter
- Example:
  - Given
    - **Parent(x,y)** : "x is a parent of y"
    - **Child(x,y)** : "x is a child of y"
  - $\forall x\ \forall y$ (Parent(x,y) $\rightarrow$ Child(y,x))
  - $\forall y\ \forall x$ (Parent(x,y) $\rightarrow$ Child(y,x))
  - Two equivalent ways to represent the statement:
    - For all x and y, if x is a parent of y, y is a child of x

Nested Quantifiers: Example 1

- Let domain be the real numbers,
- **P(x,y): "xy = 0"**
- Which one(s) is correct?
  - $\forall x \forall y\, P(x, y)$ ✗
  - $\exists x \exists y\, P(x, y)$ ✓
  - $\forall x \exists y\, P(x, y)$ ✓
    e.g. y = 0
  - $\exists x \forall y\, P(x, y)$ ✓
    e.g. x = 0

Nested Quantifiers: Example 2

- Translate the statement

$$\forall x\, (C(x) \wedge \underbrace{\exists y\, (C(y) \wedge F(x,y))}) $$

into English, where
  - C(x) is "x has a computer",
  - F(x,y) is "x and y are friends" and
  - the universe of discourse for both x and y is the set of all students in your school

Every student in your school has a computer and has a friend who has a computer.

Nested Quantifiers: Example 3

- Translate the statement
  "If a person is female and is a parent, then this person is someone's mother"
  as a logical expression
- Let
  - F(x): x is female
  - P(x): x is a parent
  - M(x,y): x is y's mother
- The domain is the set of all people

At least one y

$$(F(x) \land P(x)) \rightarrow M(x, y)$$

All x

$$\forall x\ ((F(x) \land P(x)) \rightarrow \exists y\ M(x, y)), \text{ or}$$

$$\forall x\ \exists y\ ((F(x) \land P(x)) \rightarrow M(x, y))$$

☺ Small Exercise ☺

- Translating the following statement into logic expression:

  "The sum of the two positive integers is always positive"

  - $\forall x\ \forall y\ (x+y > 0)$
    The domain for two variables consists of all positive integers
  - $\forall x\ \forall y\ ((x>0) \land (y>0) \rightarrow (x+y > 0))$
    The domain for two variables consists of all integers

☺ Small Exercise ☺

- Q(x, y, z) be the statement "x + y = z"
- The domain of all variables consists of all real
- What are the meaning of the following statements?
  - $\forall x \, \forall y \, \exists z \, Q(x,y,z)$ ✓
    - For all real numbers x and for all real numbers y there is a real number z such that x + y = z
  - $\exists z \, \forall x \, \forall y \, Q(x,y,z)$ ✗
    - There is a real number z such that for all real numbers x and for all real numbers y it is true that x + y = z

☺ Small Exercise ☺

- Translate the statement

  $\exists x \, \forall y \, \forall z$

  $((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$

  into English, where
  - F(a,b) means a and b are friends and
  - the universe of discourse for x, y and z is the set of all students in your school

There is a student none of whose friends are also friends each other

Exactly One

- It also called **uniqueness quantification** of $P(x)$ is the proposition "There exists a unique x such that the predicate is true"
- In the book, the notation is: $\exists! \, xP(x)$, $\exists_1 \, xP(x)$
- But we will try to express the concept of "exactly one" using the Universal and Existential quantifiers
- In next few slides, we assume L(x, y) be the statement "x loves y"
- Four cases will be discussed

L(x, y) : "x loves y"

Exactly One: Case 1

- Mary loves exactly one person
- It means…
  - Mary loves one person (x) $\exists x \, L(Mary, x)$
  - If any people who is not x, Mary must not love him/her $\forall z \, ((z \neq x) \rightarrow \neg L(Mary, z))$

$$\exists x \, (L(Mary, x) \land \forall z \, ((z \neq x) \rightarrow \neg \, L(Mary, z)))$$

Nested Quantifiers

L(x, y) : "x loves y"

Exactly One: Case 1 (v2)

- Mary loves exactly one person
- It means…
  - Mary loves one person (x)
  - If Mary must love any person, he/she must be x

$\exists x\, L(Mary, x)$

$\forall z\, (\, L(Mary, z) \rightarrow (z = x)\,)$

$$\exists x\, (\, L(Mary, x) \wedge \forall z\, (\, L(Mary, z) \rightarrow (z = x)\,)\,)$$

L(x, y) : "x loves y"

Exactly One: Case 1

- Mary loves exactly one person
- **Version 1**

  $\neg p \rightarrow \neg q$

  $$\exists x\, (L(Mary, x) \wedge \forall z\, ((z \neq x) \rightarrow \neg L(Mary, z))\,)$$

- **Version 2**

  $q \rightarrow p$

  $$\exists x\, (\, L(Mary, x) \wedge \forall z\, (\, L(Mary, z) \rightarrow (z = x)\,)\,)$$

- As $p \rightarrow q$ and its Contrapositive are equivalent, Version 1 and 2 are the same

L(x, y) : "x loves y"

Exactly One: Case 2

- Exactly one person loves Mary
- It means…
  - One person (x) loves Mary — $\exists x\ L(x, Mary)$
  - If anyone loves Mary, he/she must be x

$$\forall z\ (L(z, Mary) \rightarrow (z = x))$$

$$\exists x\ (L(x, Mary) \land \forall z\ (L(z, Mary) \rightarrow (z = x)))$$

L(x, y) : "x loves y"

Exactly One: Case 3

- All people love exactly one person
- It means…
  - Everyone (y) loves a person (x) — $\forall y\ \exists x\ L(y, x)$
  - If y loves someone, it must be x

$$\forall z\ (L(y, z) \rightarrow (z = x))$$

$$\forall y\ \exists x\ (L(y, x) \land \forall z\ (L(y, z) \rightarrow (z = x)))$$

Exactly One: Case 4

L(x, y) : "x loves y"

- Exactly one person loves all people
- It means…
  - A person (x) loves everyone (y) $\exists x\ \forall y\ L(x, y)$
  - ~~If anyone loves y, it must be x~~
  - If anyone loves all people, it must be x

$$\forall z\ (\ \forall w\ L(z, w) \rightarrow (z = x)\ )$$

$$\exists x\ \forall y\ (\ L(x, y)\ \land \forall z\ (\forall w\ L(z, w) \rightarrow (z = x)\ )\ )$$

Exactly One: Case 4

- Exactly one person loves all people

$$\exists x\ \forall y\ (\ L(y, x)\ \land\ \forall z\ (\boxed{\forall w\ L(z, w)} \rightarrow (z = x)\ ))$$

- Is the following answer also correct?

$$\exists x\ \forall y\ (\ L(y, x)\ \land\ \forall z\ (\boxed{L(z, y)} \rightarrow (z = x)\ ))$$

Exactly One: Case 4

$\exists x\ \forall y\ (\ L(y, x) \wedge \forall z\ (\ L(z, y) \rightarrow (z = x)\ ))$

$\forall x\ (P(x) \wedge Q(x)) \equiv \forall x\ P(x) \wedge \forall x\ Q(x)$

$\Leftrightarrow \exists x\ (\forall y\ L(y, x) \wedge \forall y\ \forall z\ (\ L(z, y) \rightarrow (z = x)\ ))$

$\Leftrightarrow \exists x\ (\forall y\ L(y, x) \wedge \forall z\ \forall y\ (\ L(z, y) \rightarrow (z = x)\ ))$

$\Leftrightarrow \exists x\ (\forall y\ L(y, x) \wedge \forall z\ \forall w\ (\ L(z, w) \rightarrow (z = x)\ ))$

$\exists x\ \forall y\ (\ L(y, x) \wedge \forall z\ (\forall w\ L(z, w) \rightarrow (z = x)\ ))$

$\forall x\ (A \wedge P(x)) \equiv A \wedge \forall x\ P(x)$

$\Leftrightarrow \exists x\ (\ \forall y\ L(y, x) \wedge \forall z\ (\forall w\ L(z, w) \rightarrow (z = x)\ ))$

Exactly One: Case 4

$\exists x\ (\ \forall y\ L(y, x) \wedge \forall z\ \forall w\ (\ L(z, w) \rightarrow (z = x)\ ))$

$\exists x\ (\ \forall y\ L(y, x) \wedge \forall z\ (\forall w\ L(z, w)\ \rightarrow (z = x)\ ))$

P(x): x works hard

A: China is great

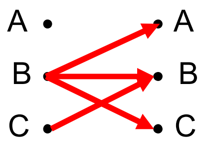
- Are they the same? **No!**
  - **$\forall x\ (P(x) \rightarrow A)$**
    - For all people, if he/she works hard, China is great
    - Any people works hard will make China great
  - **$\forall x\ (P(x)) \rightarrow A$**
    - if all people work hard, China is great
  - Therefore, $\forall x\ (P(x) \rightarrow \mathbf{A}) \equiv \exists x\ P(x) \rightarrow \mathbf{A}$

Exactly One: Case 4

$\exists x\ (\ \forall y\ L(y, x) \wedge \forall z\ \forall w\ (\ L(z, w) \rightarrow (z = x)\ ))$

$\exists x\ (\ \forall y\ L(y, x) \wedge \forall z\ (\forall w\ L(z, w)\ \rightarrow (z = x)\ ))$

- Are they the same? **No!**
  - $\forall z\ (\ \forall w\ (L(z, w) \rightarrow (z = x)\ )\ )$
    - For any people (z) and any people (w), if z is loved by w, z is x
  - $\forall z\ (\ \forall w\ (L(z, w)) \rightarrow (z = x)\ )$
    - For anyone (z), if z is loved by all people (all w), z is x



x is B
B is only person
loved by all people

L(x, y) : "x loves y"

Exactly One: Case 1 VS Case 3

- **Case 1**: Mary loves exactly one person

$\exists x\ (\ L(Mary, x) \wedge \forall z\ (\ L(Mary, z) \rightarrow (z = x)\ )\ )$

- **Case 3**: All people love exactly one person

$\forall y\ \exists x\ (\ L(y, x) \wedge \forall z\ (\ L(y, z) \rightarrow (z = x)\ )\ )$

L(x, y) : "x loves y"

Exactly One: Case 2 VS Case 4

- **Case 2**: Exactly one person loves Mary

$$\exists x\ (L(x, Mary) \wedge \forall z\ (L(z, Mary) \rightarrow (z = x)))$$

- **Case 4**: Exactly one person loves all people

$$\exists x\ \forall y\ (L(x, y) \wedge \forall z\ (\forall w\ L(z, w) \rightarrow (z = x)))$$

L(x, y) : "x loves y"

☺ Small Exercise ☺

- There is exactly one person whom everybody loves
- It means…
  - A person is loved by everyone $\exists x\ \forall y\ L(y, x)$
  - If anyone is loved by everyone, it must be x

$$\forall z\ (\forall w\ L(w, z) \rightarrow (z = x))$$

$$\exists x\ \forall y\ (L(y, x) \wedge \forall z\ (\forall w\ L(w, z) \rightarrow (z = x)))$$

L(x, y) : "x loves y"

☺ Small Exercise ☺

- Exactly two people love Mary
- It means… $\exists x \, \exists y \, (L(x, Mary) \land L(y, Mary) \land (x \neq y))$
  - At least two persons love Mary
  - At most two persons love Mary
    - If anyone loves Mary, he/she must be x or y

$$\forall z \, (L(z, Mary) \rightarrow ((z = x) \lor (z = y)))$$

$$\exists x \, \exists y \, (L(x, Mary) \land L(y, Mary) \land (x \neq y) \land \forall z \, (L(z, Mary) \rightarrow ((z = x) \lor (z = y))))$$

Nested Quantifiers

- Recall,
  - When all of the elements in the universe of discourse **can be listed one by one** (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\forall x \, P(x) \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

$$\exists x \, P(x) \equiv P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)$$

Nested Quan

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

- Example
  - Find an expression equivalent to

$$\forall x\, \exists y\, P(x, y)$$

where the universe of discourse consists of the positive integer not exceeding 3?

$$\begin{aligned}\forall x \exists y P(x, y) &= \forall x\, (\exists y P(x, y)) \\ &= \exists y P(1, y) \wedge \exists y P(2, y) \wedge \exists y P(3, y) \\ &= [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge \\ &\quad\ [P(2,1) \vee P(2,2) \vee P(2,3)] \wedge \\ &\quad\ [P(3,1) \vee P(3,2) \vee P(3,3)]\end{aligned}$$

Negating Nested Quantifiers

- Recall, De Morgan's Laws for Quantifiers
  - $\neg \forall x\, P(x) \equiv$
  - $\neg \exists x\, P(x) \equiv$
- They also can be applied in Nested Quantifiers

Negating Nested Quantifiers

- Example:
  - What is the negation of $\forall x \, \exists y \, (xy = 1)$?
  - $\neg \forall x \, \exists y \, (xy = 1) \quad = \neg \forall x \, (\exists y \, (xy = 1))$

Not every x, there are some y, can make “xy=1” success

$= \exists x \, (\neg \exists y \, (xy = 1))$

$= \exists x \, (\forall y \, \neg (xy = 1))$

$= \exists x \, (\forall y \, (xy \neq 1))$

Some x, for all y, cannot make “xy = 1” success

$= \exists x \forall y \, (xy \neq 1)$

Can you understand it now?