Chapter 1: Logic and Proof

1.1 Propositional Logic

1.2 Propositional Equivalences

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Agenda

- Ch1.1 Propositional Logic
  - Proposition
  - Propositional Operator
  - Compound Proposition
  - Applications
- Ch1.2 Propositional Equivalences
  - Logical Equivalences
  - Using De Morgan's Laws
  - Constructing New Logical Equivalences

Warm Up…

- John is a cop. John knows first aid. Therefore, all cops know first aid

Warm Up…

- Human walks by two legs. Human is mammal. Mammal walks by two legs.

Warm Up…

- The clock alarm of my iphone does not work today. The clock alarm of iphone does not work on 1-1-2011. So, today is 1-1-2011

Warm Up…

- Some students work hard to study. Some students fail in examination. So, some work hard students fail in examination.

Small Quiz

- Next few pages contain 4 questions
- Write down the answer of each question on a paper
- Remember
  - **No Discussion**
  - **Do not modify** answers you written down

Small Quiz: Question 1

- According to the law, only a person who is elder than 21-year-old can have alcoholic drink
- You are a police. Which person(s) you need to check?

Drink Tea **Drink Beer** **23-year-old** **19-year-old**

Small Quiz: Question 2

- According to a policy of a company, if someone surf the Internet longer than 2 hours, he/she has to earn more than 300k
- You are the boss of this company. Which staff(s) you need to check?

1h Surfing **3h Surfing** **Earned 200k** **Earned 400k**

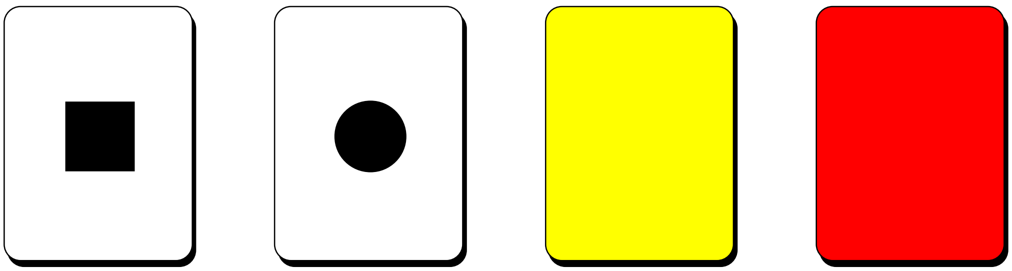
Small Quiz: Question 3

- A company publishes a desk
  Each card has two sides: a character and a number
- If one side of a card is a vowel, the number on the other side should be even number
- You are a QC staff. Which card(s) you need to check?

E **T** **4** **7**

Small Quiz: Question 4

- A company publishes another desk: each card has two sides: a shape and a color
- If one side of a card is a circle, the color on the other side should be yellow
- You are a QC staff. Which card(s) you need to check?



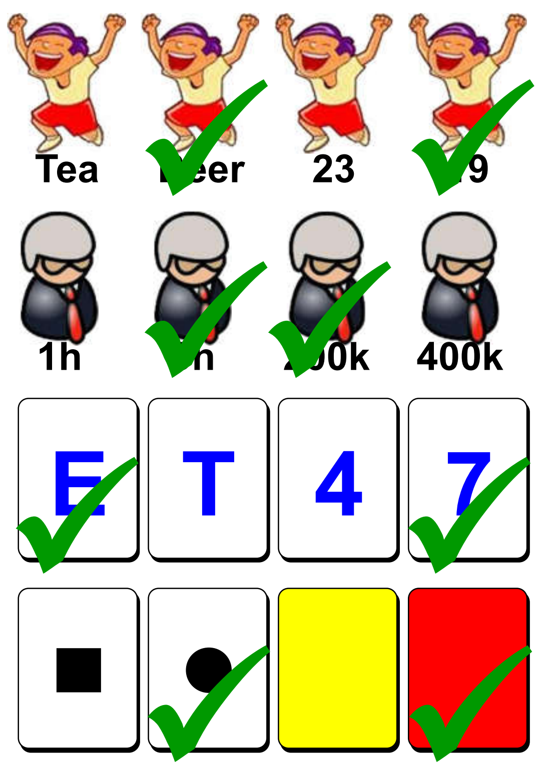
Small Game: Answer

Q1: Only a person who is elder than 21-year-old can have alcoholic drink

Q2: If someone surf the Internet longer than 2 hours, he/she has to earn more than 300k

Q3: If one side of a card is a vowel, the number on the other side is even number

Q4: If one side of a card is a circle, the color on the other side is yellow



Introduction

- In this chapter, we will explain how to
  - make up a correct mathematical argument
  - prove the arguments

Propositions

- **Proposition** (also called **statement**) is a *declarative sentence* (declares a fact) that is either true or false, but not both
- **Truth value** of a proposition is either True/False (T/F) to indicate its correctness
- Example:
  - Keep quite ✗ Not declarative
  - 1 hour has 50 minutes ✓ False
  - 1 + 1 = 3 ✓ False
  - x + 2 = 4 ✗ Can be either true or false
    Can be turned into proposition when x is defined

Propositions

- **Proposition Variable** is letters denote propositions
  - Conventional letters are $p,q,r,s,\ldots\ldots P,Q\ ,\ldots\ldots$
  - Example: r : Peter is a boy
- **Proposition Logic** is the area of logic that deals with propositions
- **Logic Operators**
  - NOT
  - AND
  - OR
  - XOR
  - If… then (Conditional Statement)
  - If and Only If (Biconditional Statement)

Proposition Logic

Negation Operator (Not)

- Definition
  - Let ***p*** be a proposition
  - Negation of *p* is the statement "It is not the case that ***p***"
  - Notation: $\neg p$, $\sim p$, $\overline{p}$
    - Read as "not *p*"
- Truth value
  - **Opposite** of the truth value of ***p***
- Example:
  - ***p***: you are a student
  - $\neg p$: You are **not** a student

Proposition Logic
Conjunction Operator (AND)

- Definition
  - Let ***p*** and ***q*** be propositions
  - Conjunction of *p* and *q* is "***p*** and ***q***"
  - Notation: ***p*** ∧ ***q***
    - ∧ points up like an "A", which means "∧ND"
- Truth value
  - **True** when both ***p*** and ***q*** are true
  - **False** otherwise
- Example:
  - ***p***: Peter likes to play, ***q***: Peter likes to read
  - ***p*** ∧ ***q*** : Peter likes to play **and** Peter likes to read

Proposition Logic
Disjunction Operator (OR)

- Definition
  - Let ***p*** and ***q*** be propositions
  - Disjunction of *p* and *q* is "***p*** or ***q***"
  - Notation: ***p*** ∨ ***q***
    - ∨ points up like an "r", means "O∨"
- Truth value
  - **False** when both ***p*** and ***q*** are false
  - **True** otherwise
- Example:
  - ***p***: Peter likes to play, ***q***: Peter likes to read
  - ***p*** ∨ ***q*** : Peter likes to play **or** Peter likes to read

Disjunction Operator (OR)

- In English, **OR** has *more than one meanings*
- Example:
  - Jackie is a singer **OR** Jackie is an actor
    - Either one or both (***inclusive***)
    - **Disjunction operation** (**OR**, $\vee$)
  - Jackie is a man **OR** Jackie is a woman
    - Either one but no both (***exclusive***)
    - **Exclusive OR operation** ($\oplus$)

Exclusive OR Operator (XOR)

- Definition
  - Let p and q be propositions
  - Notation: $p \oplus q$, $p \neq q$, $p + q$
- Truth value
  - **True** when exactly one of p and q is true
  - **False** otherwise
- Example:
  - p: You can have a tea, q: You can have a coffee
  - $p \oplus q$: You can have a tea or a coffee, but not both (exclusive or)

☺ Small Exercise ☺

- Given

  p: "Today is Friday" q: "It is raining today"

- What is…?
  - $\neg p$

    Today is not Friday

    Which is correct? Why?

    Tomorrow is Wednesday ✗

    Yesterday is Friday ✗

    Today is not Monday ✗

    They provide more information than "$\neg p$"
  - $p \wedge q$

    Today is Friday and it is raining today
  - $p \vee q$

    Today is Friday or it is raining today
  - $p \oplus q$

    Either today is Friday or it is raining today, but not both

☺ Small Exercise ☺

- Given

  p: "x > 50" q: "x < 100"

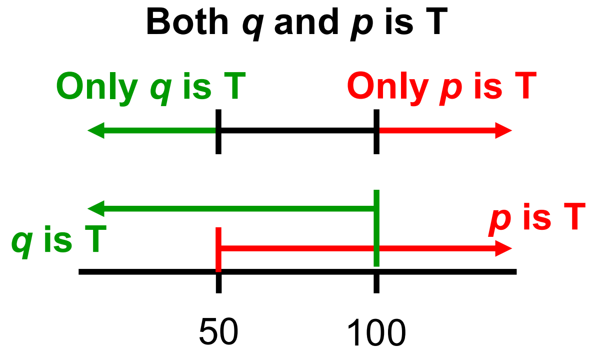
- What is…?
  - $\neg p$

    $x \leq 50$
  - $p \wedge q$

    $100 > x > 50$
  - $p \vee q$

    x can be any number
  - $p \oplus q$

    $x \geq 100$ or $x \leq 50$



Proposition Logic

Truth Table

- **Truth Table** displays the relationships between the truth values of propositions
- **Example:**
  - **Truth Table of Negation Operation**

P	¬P
T	F
F	T

Operand Column — Result Column

Proposition Logic

Truth Table

		NOT	AND	OR	XOR
p	q	¬p	p ∧ q	p ∨ q	p ⊕ q
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

Proposition Logic

Conditional Statement (imply)

- Definition
  - Let ***p*** and ***q*** be propositions
  - Conditional statement is "if *p*, then *q*"
  - Notation: $p \to q$
  - ***p*** is called the *hypothesis* (or antecedent or premise)
  - ***q*** is called the *conclusion* (or consequence)
- Truth value
  - **False** when ***p*** is true and ***q*** is false
  - **True** otherwise

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Example
  - ***p***: you work hard, ***q***: you will pass this subject
  - $p \to q$: If you work hard, then you will pass this subject

Proposition Logic

Conditional Statement (imply)

- Example:
  - *p*: "You give me twenty dollars"
  - *q*: "We are the best friends"
  - What is $p \to q$?
    - If you give me twenty dollars, then we are the best friends
  - Assume $p \to q$ is true, what does "you do not give me twenty dollars" ($\neg p$) mean?
    - Does it mean "We are not the best friend" ($\neg p \to \neg q$)? ✘

Proposition Logic

Conditional Statement (imply)

- Example:

$p \rightarrow q$ and its Contrapositive are *equivalent*

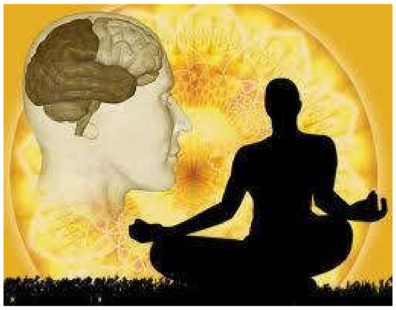
Converse and Inverse are equivalent

  - Given $p \rightarrow q$
    "If it rains, the floor is wet"
  - Situation 1 $(\neg p \rightarrow \neg q)$ **Inverse**
    If it does not rain, the floor is not wet
  - Situation 2 $(q \rightarrow p)$ **Converse**
    If the floor is wet, it rains
  - Situation 3 $(\neg q \rightarrow \neg p)$ **Contrapositive**
    If the floor is not wet, it does not rain

Proposition Logic: Conditional Statement

Necessary Condition

- To say that p is a ***necessary*** *condition* for q, it is impossible to have q without p
  - Example
    - Breathing is necessary condition for human life
      - You cannot find a non-breathing human who is alive
    - Taking a flight is not necessary condition to go to Beijing
      - You can go to Beijing by train, bus…



Proposition Logic: Conditional Statement

Sufficient Condition

- To say that p is a ***sufficient*** condition for q, the presence of p guarantees the presence of q
  - Example
    - Being divisible by 4 is sufficient for being an even number

    - Working hard is not sufficient for having a good examination result

Proposition Logic: Conditional Statement

Necessary / Sufficient Condition

- Relation between conditional statement and necessary / sufficient condition
  - **Necessary Condition**
    - E.g. Breathing is necessary condition for human life

P	Q	P is necessary condition of Q
T	T	T
T	F	T
F	T	F
F	F	T

  - **Sufficient Condition**
    - E.g. Being divisible by 4 is sufficient for being an even number

P	Q	P is sufficient condition of Q
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- $p \rightarrow q$ is equivalent to:
  - p is sufficient condition of q
  - q is necessary condition of p

Proposition Logic
Conditional Statement (imply)

- Other equivalent forms for **P** $\rightarrow$ **Q**:
  - P is a sufficient condition for Q
  - Q is a necessary condition for P
  - P implies Q
  - If P, then Q
  - If P, Q
  - Q if P
  - Q whenever P
  - P only if Q

P cannot be true when Q is not true
Q is necessary condition for P

Proposition Logic
Conditional Statement (imply)

- Remark:
  - No causality is implied in P $\rightarrow$ Q
    - P may not cause Q
  - For example:
    - If I have more money than Bill Gates, then a rabbit lives on the moon

Conditional Statement (imply)

- Example:
  - A mother tells her child that "**If you finish your homework, then you can eat the ice-cream**"
  - What does it mean?

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

  - **Case 1 ($p \rightarrow q$)**
    - Homework is finished, you can eat the ice-cream
    - Homework is not finished, you can/cannot eat the ice-cream
  - **Case 2**
    - Homework is finished, you can eat the ice-cream
    - Homework is not finished, you cannot eat the ice-cream

Biconditional Statement (equivalent)

- Definition
  - Let ***p*** and ***q*** be propositions
  - Biconditional statement is "*p* if and only if *q*" (iff)
  - Notation: $p \leftrightarrow q$, $p = q$, $p \equiv q$
  - Also called bi-implications, equivalence
  - Equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$
- Truth value
  - **True** when ***p*** and ***q*** have the same truth values
  - **False** otherwise

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Statement (equivalent)

- Example:
  - *p*: "You take the flight"
  - *q*: "you buy a ticket"
  - What is $p \leftrightarrow q$?
    - You take the flight if and only if you buy a ticket
      - No ticket, no flight
      - No flight, no ticket

Necessary / Sufficient Condition

- p is necessary but not sufficient for q

$$q \rightarrow p$$

- p is sufficient but not necessary for q

$$p \rightarrow q$$

- p is both necessary and sufficient for q

$$q \rightarrow p \wedge p \rightarrow q \qquad p \leftrightarrow q$$

  - q is also both necessary and sufficient for p

Proposition Logic

- Remarks:
  - In ordinary speech, words like "or" and "if-then" may have multiple meanings
  - In this technical subject, we assume that
    - **"or"** means inclusive or ($\vee$)
    - **"if-then"** means implication ($\rightarrow$)

Proposition Logic

- Summary

Formal Name	Nickname	Symbol
Negation Operator	NOT	$\neg$
Conjunction Operator	AND	$\wedge$
Disjunction Operator	OR	$\vee$
Exclusive-OR Operator	XOR	$\oplus$
Conditional Statement	Imply	$\rightarrow$
Biconditional Statement	Equivalent	$\leftrightarrow$

Proposition Logic

- Summary

p	q	¬p	p ∧ q	p ∨ q	p ⊕ q	p → q	p ↔ q
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Compound Proposition

- **Compound Propositions** are formed from *existing propositions* using proposition logical operators
  - Example: Beijing is the capital of China **and** 1+1=2

- How can we determine the truth values of the complicated compound propositions involving any number of propositional variables?
  - Example:
    - What is the truth value for every situations?

$$p \rightarrow \neg q \leftrightarrow s \wedge q \oplus p$$

Compound Proposition

- Precedence of Logical Operator

Precedence	Operator	
1	$\neg$	NOT
2	$\wedge$	AND
3	$\vee$ $\oplus$	OR XOR
4	$\rightarrow$	Imply
5	$\leftrightarrow$	Equivalent

- Example:
  - $p \vee q \wedge r$
    - $p \vee (q \wedge r)$ ✓
    - $(p \vee q) \wedge r$
  - $\neg s \wedge f$
    - $(\neg s) \wedge f$ ✓
    - $\neg (s \wedge f)$
  - $a \leftrightarrow f \rightarrow b$
    - $(a \leftrightarrow f) \rightarrow b$
    - $a \leftrightarrow (f \rightarrow b)$ ✓

Compound Proposition

- Example:
  1. $p \rightarrow \neg q \leftrightarrow s \wedge q \oplus p$
  2. $p \rightarrow (\neg q) \leftrightarrow s \wedge q \oplus p$
  3. $p \rightarrow (\neg q) \leftrightarrow (s \wedge q) \oplus p$
  4. $p \rightarrow (\neg q) \leftrightarrow ((s \wedge q) \oplus p)$
  5. $(p \rightarrow (\neg q)) \leftrightarrow ((s \wedge q) \oplus p)$

Precedence	Operator
1	$\neg$
2	$\wedge$
3	$\vee$ $\oplus$
4	$\rightarrow$
5	$\leftrightarrow$

- Therefore,

$$p \rightarrow \neg q \leftrightarrow s \wedge q \oplus p$$

is equal to

$$(p \rightarrow (\neg q)) \leftrightarrow ((s \wedge q) \oplus p)$$

Compound Proposition

- Truth tables can be used to determine the truth values of the complicated compound propositions
- **Algorithm:**
  1. Write down all the combinations of the compositional variables
  2. Find the truth value of each compound expression that occurs in the compound proposition according to the operator precedence

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \to \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \rightarrow \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \rightarrow \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s	¬q
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \to \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s	¬q	s ∧ q
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	T	F
F	T	T	F	T
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \to \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s	¬q	s ∧ q	(s ∧ q) ⊕ p
T	T	T	F	T	F
T	T	F	F	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	T	F	F
F	F	F	T	F	F

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \rightarrow \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s	¬q	s ∧ q	(s ∧ q) ⊕ p	p → ¬q
T	T	T	F	T	F	F
T	T	F	F	F	T	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	F	T	T	T
F	T	F	F	F	F	T
F	F	T	T	F	F	T
F	F	F	T	F	F	T

Compound Proposition

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- Example: $(p \rightarrow \neg q) \leftrightarrow (s \wedge q) \oplus p$

p	q	s	¬q	s ∧ q	(s ∧ q) ⊕ p	p → ¬q	(p → ¬q) ↔ (s ∧ q) ⊕ p
T	T	T	F	T	F	F	T
T	T	F	F	F	T	F	F
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	T	F	F	T	F
F	F	F	T	F	F	T	F

☺ Small Exercise ☺

- Write down the truth table for the following compound statement:
- p ∨ r ∧ q ↔ p ⊕ ¬ r

p	q	¬p	p ∧ q	p ∨ q	p ⊕ q	p → q	p ↔ q
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

1	¬
2	∧
3	∨ ⊕
4	→
5	↔

☺ Small Exercise ☺

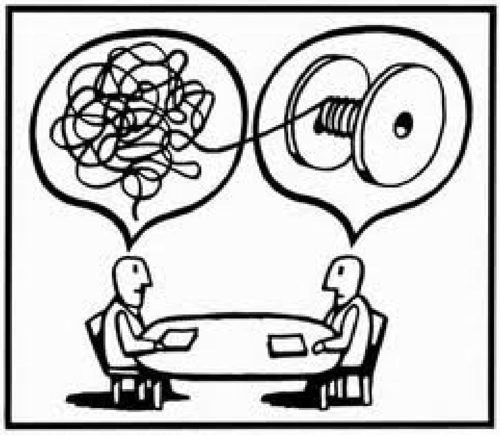
1	¬
2	∧
3	∨ ⊕
4	→
5	↔

- p ∨ r ∧ q ↔ p ⊕ ¬ r
- (p ∨ (r ∧ q)) ↔ (p ⊕ (¬ r))

p	q	r	¬r	r ∧ q	p ∨ (r ∧ q)	p ⊕ (¬ r)	(p ∨ (r ∧ q)) ↔ (p ⊕ (¬ r))
T	T	T	F	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	F	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	T	F	F
F	T	F	T	F	F	T	F
F	F	T	F	F	F	F	T
F	F	F	T	F	F	T	F

Translating English Sentences

- Human language is often **ambiguous**
- Translating human language into compound propositions (logical expression) removes the **ambiguity**



Translating English Sentences

- **Algorithm:**
  1. Remove the connective operators
  2. Let a variable for each complete concept
  3. Use the operators to connect the variables
  4. Adding brackets in suitable positions will be helpful
- Example:
  - p: "You can access the Internet from campus"
  - q: "You are a computer science major"
  - s: "You are a freshman"
  - You can access the Internet from campus only if you are a computer science major or you are not a freshman

$$p \rightarrow (q \vee \neg s)$$

System Specifications

- Specifications are the essential part of the system and software engineering
- Specifications should be consistent, otherwise, no way to develop a system that satisfies all specifications
  - Consistence means
    all specifications can be true

System Specifications

- Example:
  - There are **three specifications** for a particular system, are they **consistent**?
    - "The diagnostic message is stored in the buffer or it is retransmitted."
    - "The diagnostic message is not stored in the buffer."
    - "If the diagnostic message is stored in the buffer, then it is retransmitted."

Applications
System Specifi

1. Remove the connective operators
2. Let a variable for each complete concept
3. Use the operators to connect the variables
4. Adding brackets in suitable positions will be helpful

- "The diagnostic message is stored in the buffer or it is retransmitted." $P \vee Q$
- "The diagnostic message is not stored in the buffer." $\neg P$
- "If the diagnostic message is stored in the buffer, then it is retransmitted." $P \rightarrow Q$

- P: The diagnostic message is stored in the buffer
- Q: The diagnostic message is retransmitted

- These specifications are consistent

P	Q	$P \vee Q$	$\neg P$	$P \rightarrow Q$	$(P \vee Q) \wedge (\neg P) \wedge (P \rightarrow Q)$
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	T	T	T
F	F	F	T	T	F

Applications
Logic and Bit Operations

- Information stored in a computer is represented by bits
  - E.g. A = 0100 0001
- **Bit** = **B**inary Dig**it**, i.e. **0** or **1** (**F** or **T**)
- Logic connectives can be used as bit operation
  - **Bitwise OR ($\vee$)**
    - the OR of the corresponding bits in the two strings
  - **Bitwise AND ($\wedge$)**
    - the AND of the corresponding bits in the two strings
  - **Bitwise XOR ($\oplus$)**
    - the XOR of the corresponding bits in the two strings

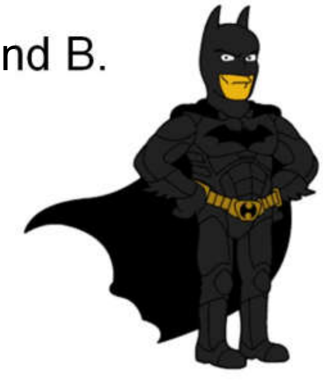
Applications
Logic and Bit Operations

- Example:

A	1011 0110
B	0001 1101
Bit-wise OR	1011 1111
Bit-wise AND	0001 0100
Bit-wise XOR	1010 1011

Applications
Logic Puzzles

- Puzzles that can be solved using logical reasoning are known as logic puzzles
- Can be solved by using rules of logic
- Example:
  - There are two kinds of people on an island
    - **Batman:** Always tell the truth
    - **Joker:** Always lie
  - One day, you encounter two peoples A and B.
    - **A** says "B is a Batman"
    - **B** says "The two of us are opposite types"
  - **What are A and B?**



Applications

Logic Puzzles

-  **Batman**: Always tell the truth
-  **Joker**: Always lie

- P — **A** says “B is a Batman”
- Q — **B** says “The two of us are opposite types”

A	B	P	Q
		T	F
		F	F
		F	T
		T	T

Types of Proposition

- **Tautology**
  - A compound proposition which is always true
  - Example: $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

- **Contradiction**
  - A compound proposition which is always false
  - Example: $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

- **Contingency**
  - A compound proposition which is neither a tautology nor a contradiction
  - Example: $P \oplus (P \wedge \neg P)$

P	$P \wedge \neg P$	$P \oplus (P \vee \neg P)$
T	F	T
F	F	F

Example

- Are they Tautology, Contradiction or Contingency?
  - $P \rightarrow P$ **Tautology**
  - $P \oplus P$ **Contradiction**
  - $P \leftrightarrow P$ **Tautology**
  - $P \rightarrow Q$ **Contingency**
  - $\neg P \vee Q$ **Contingency**
  - $\neg(P \rightarrow Q) \wedge Q$ **Contradiction**

Logically Equivalence

- An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value
- We would like to discuss about the **equivalences** of arguments

Logically Equivalence

- Definition
  Two propositions P and Q are logically equivalent if $P \leftrightarrow Q$ is a tautology
- Notation: $P \Leftrightarrow Q$ or $P \equiv Q$

Logically Equivalence

- **Truth Table** can be used to test if compositions are logically equivalent
- Example:
  if $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent?

p	q	$\sim p \vee q$	$p \rightarrow q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg p \vee q \equiv p \rightarrow q$

Logically Equivalence

- Example:
  Show $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	q∧r	p∨(q∧r)	p∨q	p∨r	(p∨q)∧(p∨r)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Logically Equivalence

- **Characteristic** of Truth Table
  - Assume n is the number of variables, Raw of tables = 2^n
    - E.g. 20 variables, $2^{20} = 1048576$
  - Not efficient

p
T
F

p	q
T	T
T	F
F	T
F	F

p	q	s
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

p	q	s	T
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

- Besides the Truth Table, we will introduce
  a series of logical equivalences

Logically Equivalence

- Example:
  - **Show $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$**

$$(p \rightarrow q) \vee (p \rightarrow r)$$
$$\equiv (\neg p \vee q) \vee (\neg p \vee r)$$
$$\equiv (\neg p \vee \neg p) \vee (q \vee r)$$
$$\equiv \neg p \vee (q \vee r)$$
$$\equiv p \rightarrow (q \vee r)$$

Logical Equivalences

$P \rightarrow Q \equiv \neg P \vee Q$

$P \vee P \equiv P$

$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

p	q	r	q∧r	p∨(q∧r)	p∨q	p∨r	(p∨q)∧(p∨r)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Important Equivalences

- **Identify Laws**

$p \wedge T \equiv p$

$p \vee F \equiv p$

p	T	p ∧ T
T	T	T
F	T	F

p	F	p ∨ F
T	F	T
F	F	F

- **Domination Laws**

$p \vee T \equiv T$

$p \wedge F \equiv F$

p	T	p ∨ T
T	T	T
F	T	T

p	F	p ∧ F
T	F	F
F	F	F

Important Equivalences

- **Idempotent Laws**

$p \vee p \equiv p$

$p \wedge p \equiv p$

p	p ∨ p
T	T
F	F

p	p ∧ p
T	T
F	F

- **Double Negation Law**

$\neg(\neg p) \equiv p$

p	¬p	¬(¬p)
T	F	T
F	T	F

Important Equivalences

- **Negation Laws**

$p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

p	¬p	p ∨ ¬p
T	F	T
F	T	T

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Important Equivalences

- **Commutative Laws**

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

p	q	q ∨ p	q ∨ p
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

p	q	q ∧ p	q ∧ p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Important Equivalences

- **Associative Laws**

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

p	q	r	q ∨ r	p ∨ (q∨r)	p ∨ q	(p∨q) ∨ r
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	F	F	F

p	q	r	q ∧ r	p ∧ (q∧r)	p ∧ q	(p∧q) ∧ r
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Important Equivalences

- Distributive Laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Important Equivalences

- How about
  - $p \vee (p \wedge q)$?
    - $(p \vee p) \wedge (p \vee q)$
    - $p \wedge (p \vee q)$
  - $p \wedge (p \vee q)$?
    - $(p \wedge p) \vee (p \wedge q)$
    - $p \vee (p \wedge q)$

Distributive Laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Important Equivalences

- **Absorption Laws**

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

p	q	p ∧ q	p ∨ (p ∧ q)
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

p	q	p ∨ q	p ∧ (p ∨ q)
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Important Equivalences

- **De Morgan's Laws**

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	p ∨ q	¬(p∨q)	¬p	¬q	¬p ∧ ¬q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

p	q	p ∧ q	¬(p∧q)	¬p	¬q	¬p ∨ ¬q
T	T	T	F	F	F	F
T	F	F	F	F	T	F
F	T	F	F	T	F	F
F	F	F	T	T	T	T

Important Equivalences

Recall, De Morgan's Laws

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

- **De Morgan's Laws Extension**
  - **$\neg(p_1 \vee p_2 \vee \ldots \vee p_n)$?**
    - Assume **q** = $p_2 \vee \ldots \vee p_n$

      $\neg(p_1 \vee p_2 \vee \ldots \vee p_n) = \neg(p_1 \vee \mathbf{q})$
    - According to De Morgan's Law

      $\neg(p_l \vee \mathbf{q}) = \neg p_1 \wedge \neg \mathbf{q} = \neg p_1 \wedge \neg(p_2 \vee \ldots \vee p_n)$

      .

      .

      .

Important Equivalences

Recall, De Morgan's Laws

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

- Assume **s** = $p_3 \vee \ldots \vee p_n$

  $\neg(p_2 \vee p_3 \vee \ldots \vee p_n) = \neg(p_2 \vee \mathbf{s})$
- According to De Morgan's Law

  $\neg(p_2 \vee \mathbf{s}) = \neg p_1 \wedge \neg \mathbf{s} = \neg p_2 \wedge \neg(p_3 \vee \ldots \vee p_n)$

  …
- Therefore,

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) = \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$$

Important Equivalences

- **De Morgan's Laws Extension**
  - Therefore,

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

$$\neg(p_1 \lor p_2 \lor \ldots \lor p_n) = \neg p_1 \land \neg p_2 \land \ldots \land \neg p_n$$

  - Similarly,

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p_1 \land p_2 \land \ldots \land p_n) = \neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$$

Identify Laws	$p \land T \equiv p$ $p \lor F \equiv p$
Domination Laws	$p \lor T \equiv T$ $p \land F \equiv F$
Idempotent Laws	$p \lor p \equiv p$ $p \land p \equiv p$
Negation Laws	$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$
Associative Laws	$p \lor (q \lor r) \equiv (p \lor q) \lor r$ $p \land (q \land r) \equiv (p \land q) \land r$
Distributive Laws	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
Absorption Laws	$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$
De Morgan's Laws	$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$

Some Important Equivalences

- Important equivalences about Implication
  - **$P \to Q \equiv \neg P \lor Q$** You only need to memorize this
  - $P \to Q \equiv \neg Q \to \neg P$
  - $P \lor Q \equiv \neg P \to Q$
  - $P \land Q \equiv \neg(P \to \neg Q)$
  - $\neg(P \to Q) \equiv P \land \neg Q$
  - $(P\to Q)\land(P\to R) \equiv P\to(Q\land R)$
  - $(P\to R)\land(Q\to R) \equiv (P\lor Q)\to R$
  - $(P\to Q)\lor(P\to R) \equiv P\to(Q\lor R)$
  - $(P\to R)\lor(Q\to R) \equiv (P\land Q)\to R$

$P \to Q \equiv \neg Q \to \neg P$

Let $P = \neg S$ and $Q = \neg T$

$P \to Q$

$\equiv \neg P \lor Q$ **

$\equiv \neg(\neg S) \lor \neg T$ *Substitution*

$\equiv S \lor \neg T$ *Double Negation Law*

$\equiv T \to S$ **

$\equiv \neg Q \to \neg P$ *Substitution*

$P \lor Q \equiv \neg P \to Q$

Let $P = \neg S$

$P \lor Q$

$\equiv \neg S \lor Q$ *Substitution*

$\equiv S \to Q$ **

$\equiv \neg P \to Q$ *Substitution*

$\neg(P \to Q) \equiv P \land \neg Q$

$\neg(P \to Q)$

$\equiv \neg(\neg P \lor Q)$ **

$\equiv P \land \neg Q$ *De Morgan's Laws*

$P \land Q \equiv \neg(P \to \neg Q)$

$P \land Q$

$\equiv \neg(\neg P \lor \neg Q)$ *De Morgan's Laws*

$\equiv \neg(P \to \neg Q)$ **

****$P\to Q \equiv \neg P\lor Q$**

$(P\to Q)\wedge(P\to R) \equiv P\to(Q\wedge R)$

$(P\to Q) \wedge (P\to R)$

$\equiv (\neg P\vee Q) \wedge (\neg P\vee R)$ **

$\equiv \neg P \vee (Q \wedge R)$ *Distributive Laws*

$\equiv P \to (Q\wedge R)$ **

$(P\to R)\wedge(Q\to R) \equiv (P\vee Q)\to P$

$(P\to R) \wedge (Q\to R)$

$\equiv (\neg P\vee R) \wedge (\neg Q\vee R)$ **

$\equiv (\neg P \wedge \neg Q) \vee R$ *Distributive Laws*

$\equiv \neg(P \vee Q) \vee R$ *De Morgan'sLaws*

$\equiv (P\vee Q) \to R$ **

$(P\to Q)\vee(P\to R) \equiv P\to(Q\vee R)$

$(P\to Q) \vee (P\to R)$

$\equiv (\neg P\vee Q) \vee (\neg P\vee R)$ **

$\equiv \neg P\vee\neg P \vee (Q\vee R)$ *Associative Laws*

$\equiv \neg P \vee (Q \vee R)$ *Idempotent Laws*

$\equiv P \to (Q\vee R)$ **

$(P\to R)\vee(Q\to R) \equiv (P\wedge Q)\to P$

$(P\to R) \vee (Q\to R)$

$\equiv (\neg P\vee R) \vee (\neg Q\vee R)$ **

$\equiv (\neg P\vee\neg Q) \vee R\vee R$ *Associative Laws*

$\equiv (\neg P\vee\neg Q) \vee R$ *Idempotent Laws*

$\equiv \neg(P \wedge Q) \vee R$ *De Morgan's Laws*

$\equiv (P\wedge Q) \to R$ **

$**P\to Q \equiv \neg P\vee Q$

Some Important Equivalences

- Important equivalences about if and only if:
  - **$P \leftrightarrow Q \equiv (P\to Q)\wedge(Q\to P)$**
  - $P \leftrightarrow Q \equiv (P\wedge Q)\vee(\neg P\wedge\neg Q)$
  - $P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$
  - $\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

You only need to memorize this

$$P \leftrightarrow Q \equiv (P\to Q)\wedge(Q\to P)$$
$$\equiv (\neg P\vee Q)\wedge(\neg Q\vee P)$$

$**P\to Q \equiv \neg P\vee Q$

Some Important Equivalences

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

** $P \rightarrow Q \equiv \neg P \vee Q$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$

$P \leftrightarrow Q$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$ ##

$\equiv ((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)$ *Distributive Laws*

$\equiv ((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)) \vee ((\neg P \wedge P) \vee (Q \wedge P))$ *Distributive Laws*

$\equiv ((\neg P \wedge \neg Q) \vee F) \vee (F \vee (Q \wedge P))$ *Negation Laws*

$\equiv (\neg P \wedge \neg Q) \vee (Q \wedge P)$ *Identify Laws*

Some Important Equivalences

$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$

** $P \rightarrow Q \equiv \neg P \vee Q$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$

Let $P = \neg S$ and $Q = \neg T$

$P \leftrightarrow Q$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$ ##

$\equiv (S \vee \neg T) \wedge (T \vee \neg S)$ *Substitution*

$\equiv S \leftrightarrow T$ ##

$\equiv \neg P \leftrightarrow \neg Q$ *Substitution*

Some Important Equivalences

** $P \rightarrow Q \equiv \neg P \vee Q$

$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$

$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

$\neg(P \leftrightarrow Q)$

$\equiv \neg((\neg P \vee Q) \wedge (\neg Q \vee P))$ ##

$\equiv \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)$ *De Morgan's Laws*

$\equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$ *De Morgan's Laws*

$\equiv ((P \wedge \neg Q) \vee Q) \wedge ((P \wedge \neg Q) \vee \neg P)$ *Distributive Laws*

$\equiv ((P \vee Q) \wedge (\neg Q \vee Q)) \wedge ((P \vee \neg P) \wedge (\neg Q \vee \neg P))$ *Distributive Laws*

$\equiv (P \vee Q) \wedge T \wedge T \wedge (\neg Q \vee \neg P)$ *Negation Laws*

$\equiv (P \vee Q) \wedge (\neg Q \vee \neg P)$ *Identify Laws*

$\equiv P \leftrightarrow \neg Q$ ##