Discrete Mathematics

Tutorial 11 - Answer

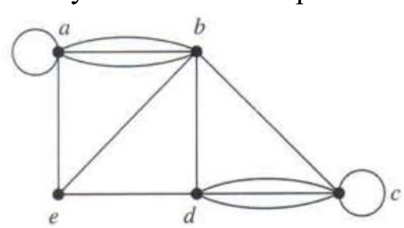
Refer to Chapter 9.1, 9.2, 9.3 and 9.4

1. What kind of graph can be used to model a highway system between major cities where
   a) there is an edge between the vertices representing cities if there is an interstate highway between them?
   b) there is an edge between the vertices representing cities for each interstate highway between them?
   c) there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

Answer:

a) A simple graph would be the model here, since there are no parallel edges or loops, and the edges are undirected.
b) A multigraph would, in theory, be needed here, since there may be more than one interstate highway between the same pair of cities.
c) A pseudograph is needed here, to allow for loops.

2. Find the number of vertices, the number of edges and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



Answer:

There are 5 vertices and 13 edges.
deg(a) = 6, deg(b) = 6, deg(c) = 6, deg(d) = 5, and deg(e) = 3.
There are no isolated or pendant vertices.

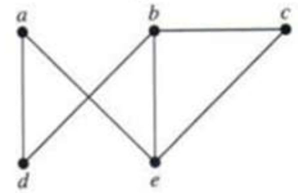
3. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) i) a, e, b, c, b

ii) a, e, a, d, b, c, a

iii) e, b, a, d, b, e

iv) c, b, d, a, e, c

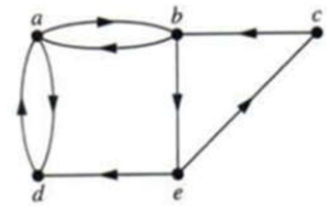


b) i) a, b, e, c, b

ii) a, d, a, d, a

iii) a, d, b, e, a

iv) a, b, e, c, b, d, a



Answer:

a)

i) Path of length 4; not a circuit; not simple.

ii) Not a path, since there is no edge between a and c.

iii) Not a path, since there is no edge between a and b.

iv) Simple circuit of length 5.

b)

i) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges ate repeated.

ii) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

iii) This is not a path, since there is no edge from d to b.

iv) There is not a path, since there is no edge from b to d.

4. Can a simple graph exist 15 vertices of degree five?

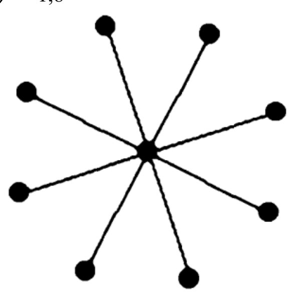
Answer:

No, because $15 \times 5 = 75$, the sum of the degrees of the vertices cannot be odd.

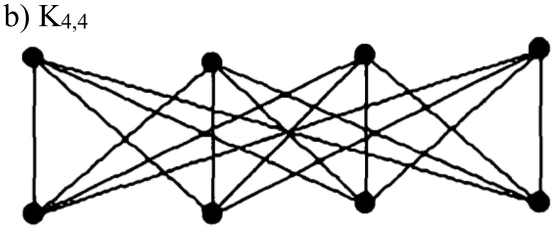
5. Draw the following graphs.
   a) $K_{1,8}$ b) $K_{4,4}$ c) Q_4

Answer:

a) $K_{1,8}$

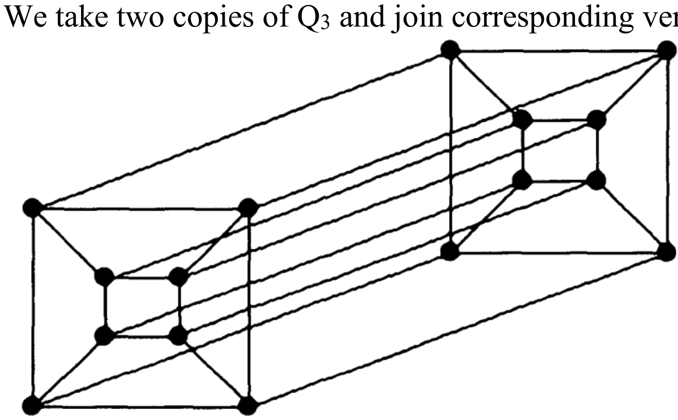


b) $K_{4,4}$

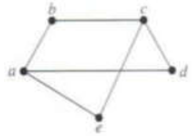


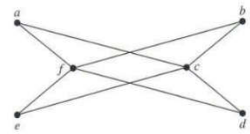
c) Q_4

We take two copies of Q_3 and join corresponding vertices.



6. Determine whether the graph is bipartite. You may answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

a) 

b) 

c) K_n

d) C_n

e) W_n

f) Q_n

Answer:

a) This graph is bipartite, with bipartition {a, c} and {b, d, e}. We can assign red or blue color to {a, c}, and the other color to {b, d, e}.

b) This graph is bipartite, with bipartition {c,f} and {a,b,d,e}.

c) By the definition given in the text, K1 does not have enough vertices to be bipartite. Clearly K_2 is bipartite. There is a triangle in K_n for n>2, so those complete graphs are not bipartite.

d) First we need n>=3 for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

e) Every wheel contains triangles, so no W_n is bipartite.

f) Q_n is bipartite for all n>=1, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

7. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$.

b) $2e/v \leq M$.

Answer:

a) Show 2e>=vm:
(Theorem 1) 2e is the sum of the degree of the vertices.
2e cannot be less than the sum of m for each vertex, since each degree is no less than m.

b) Show 2e<=vM
(Theorem 1) 2e is the sum of the degree of the vertices.
2e cannot exceed the sum of M for each vertex, since each degree is no greater than M.

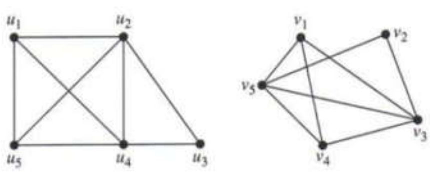
8. The complementary graph $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

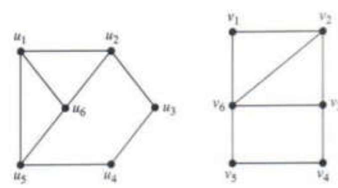
a) $\overline{K_n}$ b) $\overline{K_{m,n}}$ c) $\overline{C_n}$ d) $\overline{Q_n}$

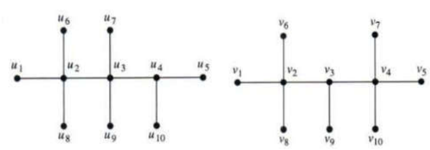
Answer:

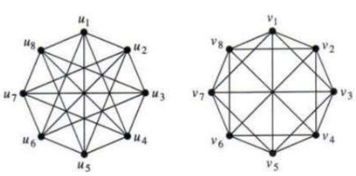
a) The graph with n vertices and no edges.
b) The disjoint union of K_m and K_n.
c) The graph with vertices $\{v_1, \ldots, v_n\}$ with an edge between v_i and v_j unless $i \equiv j \pm 1 \pmod{n}$.
d) The graph whose vertices are represented by bit strings of length n with an edge between two vertices if the associated bit strings differ in more than one bit.

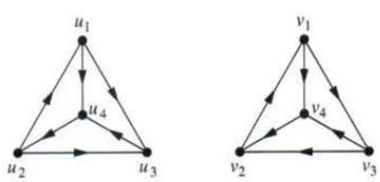
9. Determine whether the following pairs of graphs are isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

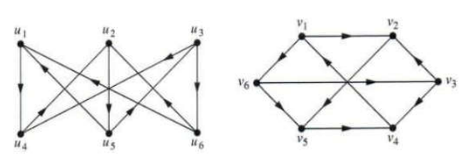
a) 

b) 

c) 

d) 

e) 

f) 

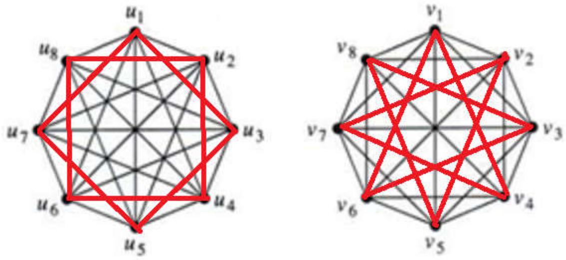
Answer:

a) Isomorphic. $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

b) Not isomorphic. Since the degrees of the vertices are not the same (the graph on the right has a vertex of degree 4, which the graph on the left lacks).

c) Not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.

d) Not isomorphic. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.



e) Isomorphic. $f(u_1) = v_3$, $f(u_2) = v_1$, $f(u_3) = v_4$, and $f(u_4) = v_2$.

f) Isomorphism.

The vertices u_1, u_2, u_3 in the first digraph are independent (i.e., have no edges joining them), as are u_4, u_5, u_6.
Similarly, The vertices v_1, v_3, v_5 and v_2, v_4, v_6 are independent in the second digraph.

Therefore these two groups of vertices will have to correspond to similar groups in the second digraph in some order.

u_3 is the only vertex among one of these groups of u's to be the only one in the group with out-degree 2, so it must correspond to v_6, the vertex with the similar property in the other digraph;
$f(u_3) = v_6$

Similarly, u_4 must correspond to v_5 as both of them with out-degree 2
$f(u_4) = v_5$

By looking at where the edges lead, pair up
u_5 with v_1 (u_3 can go to u_5; v_6 can go to v_1) $f(u_5) = v_1$
u_1 with v_2 (u_5 can go to u_1 and u_3; v_1 can go to v_2 and v_6) $f(u_1) = v_2$

u_2 with v_4 (u_4 can go to u_2; v_5 can go to v_4) $f(u_2) = v_4$
u_6 with v_3 (u_6 can go to u_2 and u_1; v_3 can go to v_4 and v_2) $f(u_6) = v_3$