Discrete Mathematics I

Tutorial 12 - Answer

Refer to Chapter 4.1, 4.2, 4.4

1. For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)
   a) $a_n = 3$
   b) $a_n = 2n$
   c) $a_n = n^2$
   d) $a_n = n^2 + n$
   e) $a_n = n + (-1)^n$
   f) $a_n = n!$

Answer:

a) $a_n = a_{n-1}, a_0 = 3$

b) $a_n = 2a_{n-1}$, $a_0 = 1$

c) $a_n = a_{n-1} + 2n - 1, a_1 = 1$

d) $a_n - a_{n-1} = 2n, a_0 = 0$

e) $a_n - a_{n-1} = 1 + 2(-1)^n, a_0 = 1$

f) $a_n = na_{n-1}, a_0 = 1$

2. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2_{n-9}$ if
   a) $a_n = -n + 2$.
   b) $a_n = 5(-l)^n - n + 2$.
   c) $a_n = 3(-1)^n + 2^n - n + 2$.
   d) $a_n = 7 \cdot 2^n - n + 2$.

Answer:

a) if $a_n = -n + 2$ then $a_{n-1} = -(n-1) + 2 = -n + 3, a_{n-2} = -(n-2) + 2 = -n + 4$

so $a_{n-1} + a_{n-2} + 2n - 9 = -n + 3 + 2(-n + 4) + 2n - 9 = -n + 2 = a_n$

b) if $a_n = 5(-1)^n - n + 2$ then $a_{n-1} = 5(-1)^{n-1} - n + 3, a_{n-2} = 5(-1)^{n-2} - n + 4$

$a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - n + 3 + 2(5(-1)^{n-2} - n + 4) + 2n - 9 = 5(-1)^n - n + 2 = a_n$

c) if $a_n = 3(-1)^n + 2^n - n + 2$

then $a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - n + 3, a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - n + 4$

$a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - n + 3 + 2(3(-1)^{n-2} + 2^{n-2} - n + 4) + 2n - 9$

$= 3(-1)^n + 2^n - n + 2 = a_n$

d) if $a_n = 7 \cdot 2^n - n + 2$ then $a_{n-1} = 7 \cdot 2^{n-1} - n + 3, a_{n-2} = 7 \cdot 2^{n-2} - n + 4$

$a_{n-1} + 2a_{n-2} + 2n - 9 = 7 \cdot 2^{n-1} - n + 3 + 2(7 \cdot 2^{n-2} - n + 4) + 2n - 9 = 7 \cdot 2^n - n + 2 = a_n$

3. a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [Hint: Express $B(k)$ in terms of $B(k - 1)$ and note that the monthly interest rate is r / 12.]
   b) Determine what the monthly payment P should be so that the loan is paid off after T months.

Answer:

a) $B(k) = B(k-1) \times (\frac{r}{12} + 1) - P$

b) let $m = \frac{r}{12} + 1$

then
$$\begin{aligned} B(k) &= B(k-1) \times m - p \\ &= (B(k-2) \times m - p) \times m - p \\ &= m^2 B(k-2) - mp - p \\ &= m^2 (mB(k-2) - p) - mp - p \\ &= .... \\ &= m^k B(0) - m^{k-1} p - ... - p \\ &= m^k B(0) - \frac{1-m^k}{1-m} p \end{aligned}$$

Let $k = T$,

If $B(T) = 0$

Then $m^T B(0) - \frac{1-m^T}{1-m} p = 0$

So $p = \frac{m^T B(0)(1-m)}{1-m^T} = \frac{(\frac{r}{12}+1)^T \cdot B(0) \cdot \frac{r}{12}}{(\frac{r}{12}+1)^T - 1}$

where $B(0)$ is the amount of original loan

4. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
   b) What are the initial conditions?
   c) How many bit strings of length seven contain two consecutive 0s?

Answer:

a) let a_n denote the number of bit strings of length n that contain a pair of consecutive 0s.

The number of bit strings of length a that contain a pair of consecutive 0s can be counted as the number of such strings of that start with a 1 plus the number of such string s that has a pair of consecutive 0s, the number is a_{n-1}. The number that start with a zero can be further broken down to the number that start 01 plus the number that start 00, the number start with 01 is a_{n-2}. Since every string that start with 00 contains a pair of consecutive zeros, this gives 2^{n-2} such strings, that is to say $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$

b) $a_0 = 0, a_1 = 0$

c) $a_7 = a_6 + a_5 + 2^5 = a_5 + a_4 + a_5 + 2^5 + 2^4$

$= 2(a_4 + a_3 + 2^3) + a_4 + 32 + 16$

$= 3(a_3 + a_2 + 2^2) + 2a_3 + 64$

$= 5a_3 + 3a_2 + 76$

$= \ldots$

$= 94$

5. a) Find a recurrence relation for the number of ways to layout a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) How many ways are there to layout a path of seven tiles as described in part (a)?

Answer:

a) $a_n = 2a_{n-1} + 2a_{n-2}$

b) $a_1 = 3, a_2 = 8$

c)

$a_7 = 2a_6 + 2a_5 = 2(2a_5 + 2a_4) + 2a_5 = 6a_5 + 4a_4$

$= 6(2a_4 + 2a_3) + 4a_4 = 16a_4 + 12a_3$

$= 16(2a_3 + 2a_2) + 12a_3 = 44a_3 + 32a_2$

$= 44(2a_2 + 2a_1) + 32a_2 = 120a_2 + 88a_1$

$= 120 \times 8 + 88 \times 3 = 1224$

6. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, -2, -2, -2, 3, 3, -4?

Answer:

$$a_n = (a_{00}n^3 + a_{01}n^2 + a_{02}n + a_{03}) + (a_{10}n^2 + a_{11}n + a_{12})(-2)^n + (a_{20}n + a_{21})3^n + a_{30}(-4)^n$$

7. Solve these recurrence relations together with the initial conditions given.
   a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
   b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
   c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
   d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
   e) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$
   f) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ for $n \geq 3$, $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$

Answer:

a) $a_n = 2a_{n-1}$

$r = 2$

so, $a_n = C \cdot 2^n$, $a_0 = C \cdot 2^0 = 3$

As a result, $a_n = 3 \cdot 2^n$

b) $a_n = a_{n-1}$

$r = 1$

so $a_n = C \cdot 1^n = C$, $a_0 = C = 2$

As a result, $a_n = 2$

c) $a_n = 5a_{n-1} + 6a_{n-2}$

$r^2 - 5r + 6 = 0, r = 2, r = 3$

So $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$

$a_0 = C_1 \cdot 2^0 + C_2 \cdot 3^0 = C_1 + C_2 = 1$

$a_1 = C_1 \cdot 2^1 + C_2 \cdot 3^1 = 2C_1 + 3C_2 = 0$

$C_1 = 3, C_2 = -2$

As a result, $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

d) $a_n = 4a_{n-1} - 4a_{n-2}$

$r^2 - 4r + 4 = 0, r = 2$

So, $a_n = 2^n(Cn + d)$

$a_0 = 2^0(0n + d) = d = 6$

$a_1 = 2^1(1n + d) = 8, c + d = 4, c = -2$

As a result, $a_n = 2^n(-2n + d)$

e) $r^2 + 4r - 5 = 0, r_1 = 1, r_2 = -5$

$a_n = C_1 1^n + C_2(-5)^n$

$a_0 = C_1 1^0 + C_2(-5)^0 = C_1 + C_2 = 2$

$a_1 = C_1 1^1 + C_2(-5)^1 = C_1 - 5C_2 = 8$

$C_1 = 3, C_2 = -1$

$a_n = 3 - (-5)^n$

f) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$r^3 + 3r^2 + 3r + 1 = 0, r = -1$

$a_n = (-1)^n(bn^2 + cn + d)$

$a_0 = (-1)^0(b0 + c0 + d) = d = 5$

$a_1 = (-1)^1(b + c + 5) = -9, b + c = 4$

$a_2 = (-1)^2(4b + 2c + 5) = 15, 4b + 2c = 10, b = 1, c = 3$

As a result, $a_n = (-1)^n(n^2 + 3n + 5)$

8. Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each receives at least two balloons.

Answer:

Because we have 10 identical balloons to given four children and each one receives at least two balloons, so the maximal balloons that one child could have is 4. So we only need to calculate the coefficient of x^{10} in the expansion of:

$(x^2 + x^3 + x^4)^4$, the coefficient of x^{10} in this product is 10, so there are 10 different ways.

9. Use generating functions to find the number of ways to choose a dozen bagels from three varieties — egg, salty and plain — if at least two bagels of each kind but no more than three salty bagels are chosen.

Answer:

Because we have 12 bagels to choose, each kind of bagels have at least two, so the maximal bagels we can take for one kind is 8. So we only need to calculate the coefficient of x^{12} in the expansion of:

$(x^2+x^3+x^4+x^5+x^6+x^7+x^8)^2(x^2+x^3)$, the coefficient of x^{12} in this product is 13, so there are 13 different ways.

10. What is the generating function for the sequence $\{c_k\}$, where c_k is the number of ways to make change for k dollars using \$1 bills, \$2 bills, \$5 bills, and \$10 bills?

Answer:

$$f(x)=(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)$$
$$=\frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})}$$

11. What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

Answer: the coefficient of x^k in the expansion

$$f(x)=(x^2+x^3+\ldots)(1+x+x^2+x^3)(x^2+x^3+x^4+x^5)$$
$$=x^4(1+x+x^2+x^3+\ldots)(1+x+x^2+x^3)^2$$
$$=\frac{x^4(1+x+x^2+x^3)^2}{1-x}$$

12. Use generating functions to solve $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

Answer:

$$a_k = 5a_{k-1} - 6a_{k-2}$$

$$Let\ \ G(x)=\sum_{k=0}^{+\infty} a_k x^k$$

$$\begin{aligned}
G(x)-5xG(x)+6x^2G(x) &= \sum_{k=0}^{+\infty} a_k x^k - 5\sum_{k=0}^{+\infty} a_k x^{k+1} + 6\sum_{k=0}^{+\infty} a_k x^{k+2} \\
&= \sum_{k=0}^{+\infty} a_k x^k - 5\sum_{k=1}^{+\infty} a_{k-1} x^k + 6\sum_{k=2}^{+\infty} a_k x^k \\
&= a_0x^0 + a_1x^1 + \sum_{k=2}^{+\infty} a_k x^k - 5a_0x^1 - 5\sum_{k=2}^{+\infty} a_{k-1}x^k + 6\sum_{k=2}^{+\infty} a_k x^k \\
&= a_0 + a_1x - 5a_0x \\
&= 6 + (30-5\times 6)x = 6
\end{aligned}$$

$$\begin{aligned}
\therefore G(x) &= \frac{6}{6x^2-5x+1} = \frac{6}{(1-2x)(1-3x)} = \frac{6}{x}\cdot\frac{x}{(1-2x)(1-3x)} = -\frac{6}{x}\cdot\left(\frac{1}{1-2x}-\frac{1}{1-3x}\right) \\
&= -\frac{6}{x}\left(\sum_{k=0}^{+\infty} 2^k x^k - \sum_{k=0}^{+\infty} 3^k x^k\right) = -\frac{6}{x}\left(1+\sum_{k=1}^{+\infty} 2^k x^k - 1 - \sum_{k=1}^{+\infty} 3^k x^k\right) \\
&= -6\sum_{k=1}^{+\infty} 2^k x^{k-1} + 6\sum_{k=1}^{+\infty} 3^k x^{k-1} = -12\sum_{k=0}^{+\infty} 2^k x^k + 18\sum_{k=0}^{+\infty} 3^k x^k \\
&= \sum_{k=0}^{+\infty} (-12\cdot 2^k + 18\cdot 3^k)x^k
\end{aligned}$$

$$\therefore\quad a_k = 18\cdot 3^k - 12\cdot 2^k$$