Discrete Mathematics I

<u>Tutorial 10 - Answer</u>

<u>Refer to Chapter 3.1, 3.2, 3.3, 3.5, 3.6</u>

1. How many positive integers between 1000 and 9999 inclusive
   a) are divisible by 9?
   b) are not divisible by either 5 or 7?
   c) are divisible by 5 but not by 7?
   d) are divisible by 5 and 7?

Answer:

a) $1000 = 112 * 9 - 8$
$9999 = 1111*9$
$1111-112+1$

b, c, d)

For 5:
$1000 = 200 * 5$
$9999 = 1999 * 5 + 4$
$1999 – 200 + 1 = 1800$

For 7:
$1000 = 143 * 7 - 1$
$9999 = 1428 * 7 + 4$
$1428 – 143 + 1 = 1286$

For 5 and 7 (35):
$1000 = 29 * 35 - 15$
$9999 = 285 * 35 + 25$
$285 – 29 + 1 = 257$

b) $9000 – (1800 + 1286 – 257)$

c) $1800 - 257$

d) 257

2. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters *, >, <, !, +, and =.
   a) How many different passwords are available for this computer system?
   b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
   c) If it takes one nanosecond for a hacker to check whether each possible password is your password, how long would it take this hacker to try every possible password?

Answer:

$26+26+10+6 = 68$

a) $68^8+68^9+68^{10}+68^{11}+68^{12}$
b) $(68^8+68^9+68^{10}+68^{11}+68^{12}) - (62^8+62^9+62^{10}+62^{11}+62^{12})$
c) $(68^8+68^9+68^{10}+68^{11}+68^{12}) \times 10^{-9}$

3. How many ways are there to seat six people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

Answer:

As the table is circular, the following six permutations are the same:

a b c d e f
b c d e f a
c d e f a b
d e f a b c
e f a b c d
f a b c d a

Therefore, ${}_6P_6/6=120$

4. Prove that if n and k are integers with $1 \leq k \leq n$, then $k \cdot {}_nC_k = n \cdot {}_{n-1}C_{k-1}$

a) using a combinatorial proof. (Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.)

b) using an algebraic proof

Answer:

a) We show that each side counts the numbers of ways to choose from a set with n elements subset with k elements and a distinguished element of that set.
For the left hand side, first choose the k-set (this can be done in ${}_nC_k$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways).
For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining k-1 elements of the subset from the remaining n-1 elements of the set (this can be done in ${}_{n-1}C_{k-1}$ ways).

b) $$k\binom{n}{k} = kC_n^k = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

$$n\binom{n-1}{k-1} = kC_{n-1}^{k-1} = n\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

5. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

Answer:

n=20, r=5

${}_{20+5-1}C_4$

6. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1 s, and five 2s?

Answer:

$n_0 = 2$

$n_1 = 3$

$n_2 = 5$

$$\frac{10!}{2!3!5!} = \frac{3628800}{2 \times 6 \times 120} = 2520$$

7. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

Answer:

Assume there is no restriction,
A B C D X A B C D X

Same site will be counted twice, 10! / (2^5), (since 5 sites have be counted twice)

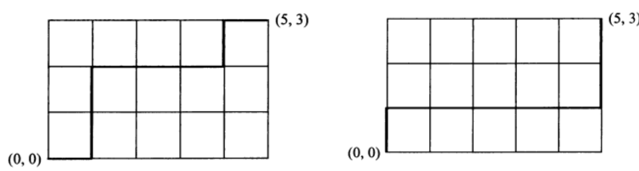
Consider the situation which visit the X site twice on the consecutive days:
A B C D XX A B C D

Same site will be counted twice, 9! / (2^4), (since 4 sites have be counted twice)

Therefore:
10! / (2^5) - 9! / (2^4)

8. In this exercise we will count the number of paths in the *xy* plane between the origin (0, 0) and point (*m*, *n*) such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.)

Two such paths from (0, 0) to (5, 3) are illustrated here.



a) Show that each path of the type described can be represented by a bit string consisting of *m* 0s and *n* 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.
b) Conclude from part (a) that there are ${}_{m+n}C_n$ paths of the desired type.

Answer:

a) A path of the desired type consists of m moves to the right and n moves up. Each such path can be represented by a bit string of length m+n with m 0s and n 1s, where a 0 represents a move to the right and a 1 a move up.

b) The number of bit strings of length m+n containing exactly n 1s equals $\binom{n+m}{n} = \binom{n+m}{m}$ because such a string is determined by specifying the positions of the n 1s or by specifying the position s of the m 0s.

9. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if
   a) both the balls and boxes are labeled?
   b) the balls are labeled, but the boxes are unlabeled?
   c) the balls are unlabeled, but the boxes are labeled?
   d) both the balls and boxes are unlabeled?

Answer:

a) ${}_7P_5$
b) 1
c) ${}_7C_5$
d) 1

10. Find the next larger permutation in lexicographic order after each of these permutations.

a) 1342
b) 45321
c) 13245
d) 612345
e) 1623547
f) 23587416

Answer:

1) 1423
2) 51234
3) 13254
4) 612354
5) 1623574
6) 23587461

11. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Answer:

Let a, a+1, …, a + n – 1 be the integers in the sequence

The integers (a+i) mod n, i = 0,1,2,…,n-1 are distinct, because 0 < (a+j) – (a+k) < n whenever 0 < k < j < n – 1

Because there are n possible values for (a+i) mod n and there are n different integers in the set, each of these values is taken on exactly once

Therefore, (a+i) mod n will equal to 0

12. How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16?

Answer:

Summation of four pairs {1, 15}, {3, 13}, {5, 11} and {7, 19} are equal to 16
If we have five numbers, at least two values are from the same pair.

13. Let $n_1, n_2, \ldots, n_t$ be positive integers. Show that if $n_l + n_2 + \ldots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \ldots, t$, the ith box contains at least n_i objects.

Answer:

Assume all boxes contain at most n_i-1 objects.
By adding the objects contained in all the boxes $n_l + n_2 + \ldots + n_t - t$, which is smaller then the number of objects given by the question.