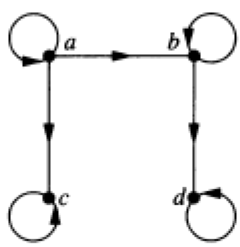
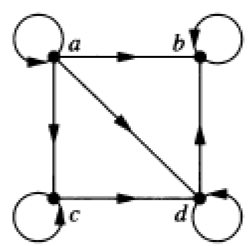
Discrete Mathematics I

Tutorial 08 - Answer

Refer to Chapter 5.6

1. Determine whether the following relations are partial order:
   a) "a is taller than b" on the set of all people in the world
   b) "a=b or a is an ancestor of b" on the set of all people in the world
   c) (R, =)
   d) (Z, ≠)
   e) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   f) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   g) 

   h) 

Answer:

a) No
b) Yes
c) Yes
d) No
e) No
f) Yes
g) No
h) No

2. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset.
   (S, R^{-1}) is called the dual of (S, R).

Answer:

As R is reflexive:
$\forall a \in S$, we have $(a,a) \in R$,
$\therefore (a,a) \in R^{-1}$
Hence, R^{-1} is reflexive.

As R is antisymmetric:
For $\forall a,b \in S$
If $(a,b) \in R$ and $(b,a) \in R$ then a=b

Give $(b,a) \in R^{-1}$ and $(a,b) \in R^{-1}$, show b=a
$(b,a) \in R^{-1}$ implies $(a,b) \in R$
$(a,b) \in R^{-1}$ implies $(b,a) \in R$
Therefore, a=b
Hence: R^{-1} is antisymmetric.

As R is transitive:
For $\forall a,b,c \in S$
If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

Give $(c,b) \in R^{-1}$ and $(\text{b},a) \in R^{-1}$, show $(c,a) \in R^{-1}$
$(c,b) \in R^{-1}$ implies $(b,c) \in R$
$(\text{b},a) \in R^{-1}$ implies $(a,b) \in R$
Therefore, $(a,c) \in R$, which implies $(c,a) \in R^{-1}$
Hence: R^{-1} is transitive.

3. The poset (S, R^{-1}) is called the dual of (S, R). Find the duals of these posets.
   a) $(\{0,1,2\}, \leq)$ b) $(Z, \geq)$

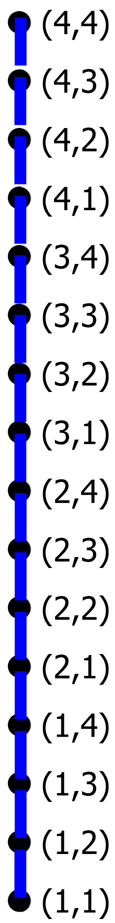
Answer:

a) {(0,0), (1,0), (1,1), (2,0), (2,1), (2,2) }
b) $(Z, \leq)$

4. Let S = {1,2,3,4}. With respect to the lexicographic order based on the usual "less than" relation,
   a) Find all pairs in S x S less than (2,3).
   b) Find all pairs in S x S greater than (3,1).
   c) Draw the Hasse diagram of the poset (S x S, ≤)

Answer:

a) (1,1), (1,2), (1,3), (1,4), (2,1), (2,2)
b) (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)
c)

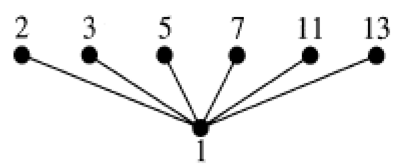


5. Find the lexicographic ordering of these strings of lower-case English letters:
   a) quack, quick, quicksilver, quicksand, quacking
   b) open, opener, opera, operand, opened
   c) zoo, zero, zoom, zoology, zoological

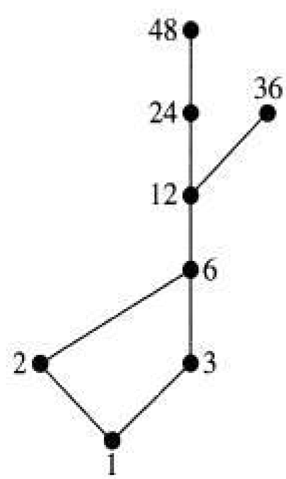
Answer:
   a) quack < quacking < quick < quicksand < quicksilver
   b) open < opened < opener < opera < operand
   c) zero < zoo < zoological < zoology < zoom

6. Draw the Hasse diagram for divisibility on the set
   a) {1,2,3,5,7,11,13}   b) {1,2,3 ,6,12,24,36,48}

Answer:
   a)

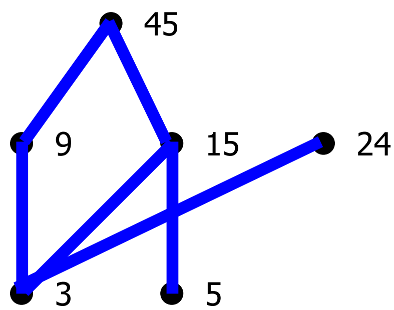


   b)



7. Answer these questions for the poset ({3, 5, 9, 15, 24, 45}, |).
   a) Find the maximal elements
   b) Find the minimal elements
   c) Is there a greatest elements?
   d) Is there a least elements?
   e) Find all upper bounds of {3, 5}.
   f) Find the least upper bound of {3, 5}, if it exists
   g) Find all lower bounds of {15, 45}
   h) Find the greatest lower bound of {15, 45}, if it exists

Answer:

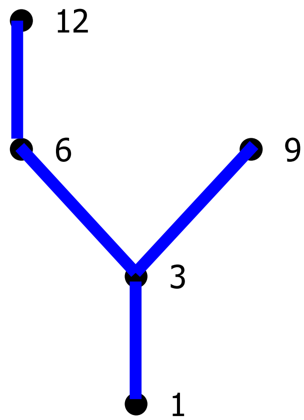


a) 24, 45
b) 3, 5
c) No
d) No
e) 15, 45
f) 15
g) 15, 5, 3
h) 15

8. Determine whether these posets are lattices.
   a) $(\{1, 3, 6, 9, 12\}, |\,)$
   b) $(Z, \geq)$
   c) $(P(S), \supseteq)$, where P(S) is the power set of a set S.

Answer:

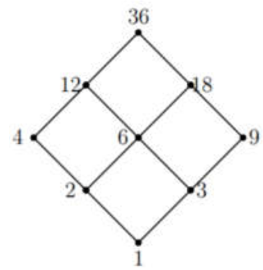
a) No (no least upper bound)



b) No (lower bound is not exist)

c) Yes

9. Given ({1, 2, 3, 4, 6, 9, 12, 18, 36}, |)
   a) Suggest the binary operations for ∨ and ∧.
   b) Show if it is distributive lattice?
   c) Show if it is a complemented lattice?



Answer:

a) ∨ = Lowest Common Multiple (LCM),
   ∧ = Greatest Common Divisor (GCD) / Highest Common Factor (HCF)

Property of HCF and LCM

1) HCF(a,b) × LCM(a,b) = a × b

24, 16
HCF(24,16) = 8
LCM(24,16) = 48

24 × 16 = 384
8 × 48 = 384

2) a = d × a_1 and b = d × b_1, where HCF(a,b) = d AND a_1 and b_1 are integer

HCF(24,16) = 8
24 = 8 × 3
16 = 8 × 2

3) HCF(a × b, a × c) = a × HCF(b, c)

HCF(24,16) = 4 × HCF(6,4) = 4 × 2 = 8

4) If b|a and c|a, then LCM(b,c)|a

2|12 and 3|12
LCM(2,3) = 6
6|12

5) If b|a and d|c, then LCM(b,d)|LCM(a,c)

3|12 and 15|30
LCM(3,15) = 15
LCM(12,30) = 60
15|60

6) If a|b and a|c, then a|HCF(b,c)

2|12 and 2|18
HCF(12,18) = 6
2|6

b) It is distributive lattice.

Prove that for all a, b, c in $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$,
there exist $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

For $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
Let $\alpha = a \wedge (b \vee c) = a \wedge \gamma$,
$\beta = (a \wedge b) \vee (a \wedge c) = \delta \vee \lambda$.

$\alpha = \beta$ if $\alpha | \beta$ and $\beta | \alpha$

Proof of $\boldsymbol{\alpha | \beta}$:
As d be HCF of b and c,
$b = b_1 * d,$ (1)
$c = c_1 * d,$ (2) Property 2 in (a)
$\gamma = b \vee c = b_1 * c_1 * d.$ (3) Property 1 in (a)

As α is HCF of a and γ,
$\gamma = \gamma_1 * \alpha,$ (4)
$a = a_1 * \alpha$ (5) Property 2 in (a)

$\alpha = \frac{\gamma}{\gamma_1} = \frac{b_1 * c_1 * d}{\gamma_1}.$ (6) (4) substituted by (3)
$a = a_1 * \alpha = \frac{a_1 * b_1 * c_1 * d}{\gamma_1}$ (7) (5) substituted by (6)

$\delta = a \wedge b$
$= \left(\frac{a_1 * b_1 * c_1 * d}{\gamma_1}\right) \wedge (b_1 * d)$ substituted by (5) & (1)
$= \frac{b_1 * d}{\gamma_1}\left((a_1 * c_1) \wedge (\gamma_1)\right)$ Property 3 in (a)
$= \frac{b_1 * d * \delta_1}{\gamma_1}$ (9)
where $\delta_1 = (a_1 * c_1) \wedge (\gamma_1)$ (10)

Similarly

$\lambda = a \wedge c$

$= \left(\frac{a_1 * b_1 * c_1 * d}{\gamma_1}\right) \wedge (c_1 * d)$ substituted by (5) & (2)

$= \frac{c_1 * d}{\gamma_1}\left((a_1 * b_1) \wedge (\gamma_1)\right)$ Property 3 in (a)

$= \frac{c_1 * d * \lambda_1}{\gamma_1}$ (11)

where $\lambda_1 = (a_1 * b_1) \wedge (\gamma_1)$ (12)

$\delta \wedge \lambda$

$= \frac{b_1 * d * \delta_1}{\gamma_1} \wedge \frac{c_1 * d * \lambda_1}{\gamma_1}$ substituted by (9) & (11)

$= \frac{d}{\gamma_1}(\delta_1 \wedge \lambda_1)$ (13) Property 3 in (a) & $b_1 \wedge c_1 = 1$ by (1)

$\beta = \delta \vee \lambda$

$= \frac{\delta * \lambda}{\delta \wedge \lambda}$ Property 1 in (a)

$= \frac{b_1 * c_1 * d * (\delta_1 * \lambda_1)}{\gamma_1 * (\delta_1 \wedge \lambda_1)}$ substituted by (9) , (11) & (13)

$= \alpha * \beta_1$ substituted by (6)

where $\beta_1 = \frac{(\delta_1 * \lambda_1)}{(\delta_1 \wedge \lambda_1)}$, β_1 is an integer

Hence $\alpha | \beta$.

Proof of $\boldsymbol{\beta | \alpha}$:

Since $\delta = (a \wedge b)$, $\delta | a$ and $\delta | b$ are true
Since $\lambda = (a \wedge c)$, $\lambda | a$ and $\lambda | c$ are true

$(\delta | a \ AND \ \lambda | a) \ AND \ (\delta | b \ AND \ \lambda | c)$

$\Longrightarrow (\delta \vee \lambda) | a \ AND \ (\delta \vee \lambda) | (b \vee c)$ Property 4 & 5 in (a)

$\Longrightarrow \beta | a \ AND \ \beta | \gamma$

$\Longrightarrow \beta | (a \wedge \gamma)$ Property 6 in (a)

$\Longrightarrow \beta | \alpha$

Hence $\beta | \alpha$

Therefore, $\alpha = \beta$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Similarly, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ is also true.

c) Not complemented lattice.
Let L = {1, 2, 3, 4, 6, 9, 12, 18, 36}
lub of L is 36, and glb of L is 1.
Some elements in L cannot find its complement.
For example, 2. No element in L such that $2 \vee y = 36$ and $2 \wedge y = 1$