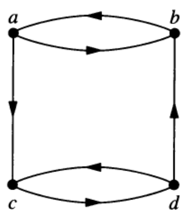
Discrete Mathematics I

Tutorial 07 - Answer

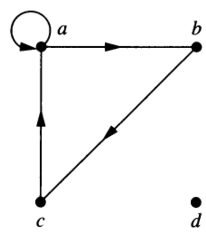
Refer to Chapter 5.4, 5.5

1. Find the directed graphs of i) the symmetric closures, ii) the reflexive closures of the relations with directed graphs shown as follows:

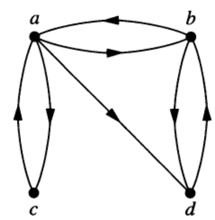
a)



b)

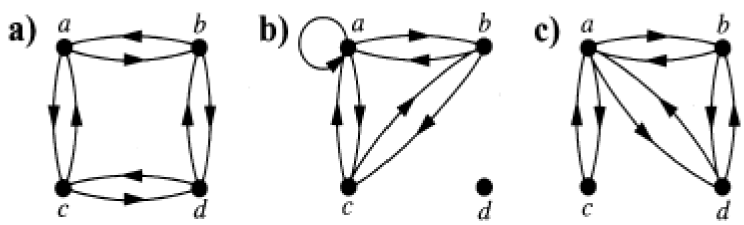


c)



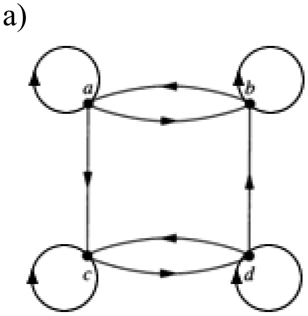
Answer：

i) the symmetric closures

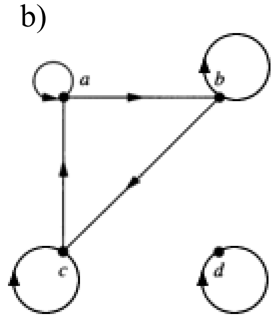


ii) the reflexive closures

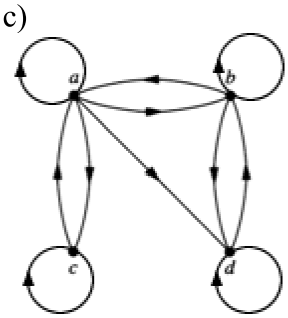
a)



b)



c)



2. Let R be the relation on the set {0,1,2,3} containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0). Find the
   a) reflexive closure of R b) symmetric closure of R c) transitive closure of R

Answer：

a) {(0,0), (0,1), (1,1), (1,2), (2,0), (2,2), (3,0), (3,3) }

b) {(0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,0) }

c) R is { (0,1), (1,1), (1,2), (2,0), (2,2), (3,0) }

Using the Warshall's algorithm ($W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee (W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]})$)

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \therefore W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \therefore W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\therefore W_3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \therefore W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

3. Find the transitive closures of these relations of {1, 2, 3, 4}.
   a) { (1,2), (2,1), (2,3), (3,4), (4,1) }
   b) { (2,1), (2,3), (3,1), (3,4), (4,1), (4,3) }
   c) { (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) }

Answer：

a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. Find the smallest relation containing the relation { (1,2), (1,4), (3,3), (4,1) } that is
   a) reflexive and transitive.
   b) symmetric and transitive.
   c) reflexive, symmetric, and transitive.

Answer：

a) {(1,1), (1,2), (1,4), (2,2), (3,3), (4,1), (4,2), (4,4) }
b) {(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,3), (4,1), (4,2), (4,4) }
c) {(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,3), (4,1), (4,2), (4,4) }

5. Suppose that the relation R is reflexive. Show that R^* is reflexive.

Answer：

For the relation R is reflexive,
So the diagonal elements of M_R are all ones.
For $M_{R^n} = (M_R)^n$
So the diagonal elements of $M_{R^2}, M_{R^3}, \ldots, M_{R^n}$ are also all ones.
That is to say,
$R^2, R^3, \ldots, R^n$ are also reflexive.
For $R^* = \bigcup_{n=1}^{\infty} R^n$
$\therefore R^*$ is also reflexive.

6. Suppose that the relation R is symmetric. Show that R^* is symmetric.

Answer：

Show R^n is symmetric for any n.

Given R is symmetric. Assume R^n is symmetric

Show R^{n+1} is symmetric
Assume (a,b) is in Rn+1, show (b,a) is also in R^{n+1}

(a,b) is in Rn+1, (a,x) in R, and (x,b) in R^n
As R is symmetric, (a,x) in R, (x,a) is also in R.
As Rn+1 is symmetric, (x,b) in R, (b,x) is also in R^n.
Therefore, since (x,a) is in R and (b,x) is in Rn, (b,a) is in R^{n+1}

Therefore, R^{n+1} is symmetric

Therefore, R^n is symmetric for any n.
Since $\bigcup_{n=1}^{\infty} R^n = R^*$, R* is symmetric

7. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

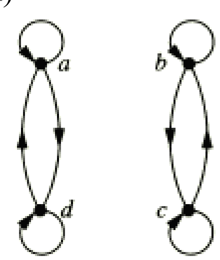
Answer：

For example

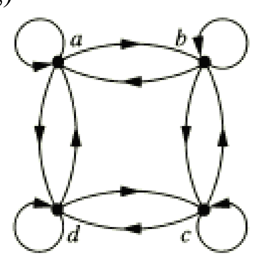
$$M_R = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ then } M_{R^2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

R is irreflexive, but R^2 is reflexive.

8. Which of these relations on {0, 1, 2, 3} are equivalence relation?
   a) { (0,0), (1,1), (2,2), (3,3) }
   b) { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)}
   c) { (0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
   d) {(a, b) | a and b have the same parents}
   e) {(a, b) | a and b have met }
   f)



   g)



   h)
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

   i)
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:
   a) Equivalence relation
   b) Not transitive
   c) Not symmetric, not transitive
   d) Yes
   e) Not transitive
   f) Yes
   g) No
   h) Yes
   i) Yes

9. Define an equivalence relation on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

Answer:
Many answer are possible.
(1) Two building are equivalent if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year (as long as there was at least one buildings opened that year).

(2) Two buildings are equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on (one class for each n for which there is at least one n-story building).

(3) Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class (including itself); there are two equivalence classes-the set of buildings in which you have a class and the set of buildings in which you don't have a class (assuming these nonempty).

10. Suppose that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three bits is an equivalence relation on the set of all bit strings of length three or more. What are the equivalence classes of these bit strings for R?

a) 010 b) 1011 c) 11111 d) 01010101

Answer:

(a) (d) are in the same equivalence relation

11.Which of these collections of subsets are partitions on the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01
b) the set of bit strings that contain the string 00, the set of bit strings contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11
c) the set of bit strings that end with 00, the set of bit strings that end with 01, and the set of bit strings that end with 10, the set of bit string that end with 11

Answer:

(a),(c)

12. Suppose that **A** is a nonempty set, and *f* is a function that has **A** as its domain. Let **R** be the relation on **A** consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on **A**.
b) What are the equivalence classes of **R**?

Answer:

a)

$(x, y) \in R$ because $f(x) = f(x)$. Hence, R is reflexive

$(x, y) \in R$ if and only if $f(x) = f(y)$, which holds if and only if and only if $f(y) = f(x)$ if and only if $(x, y) \in R$. Hence, R is symmetric.

if $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$.Hence $f(x) = f(z)$. Thus, $(x, z) \in R$. It follows that R is transitive.

b) The sets $f^{-1}(b)$ for ***b*** is in the range of ***f***.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

Answer:

Let x be a string of length 3 or more,
Because ***x*** agrees with itself in all bits, $(x,x) \in R$.
So ***R*** is Reflexive.
If ***x*** and ***y*** agree all the string bits expect the first three.
Hence, ***y*** and ***x*** also agree all the string bits expect the first three.
So ***R*** is symmetric.
If ***x*** and ***y*** agree all the string bits expect the first three.
y and ***z*** agree all the string bits expect the first three.
x and ***z*** agree all the string bits expect the first three.
So ***R*** is transitive.

14. Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.
a) $R_1 \cup R_2$ b) $R_1 \cap R_2$

Answer:

a) No

Reflexive: $\forall x \in S.(x,x) \in R_1, (x,x) \in R_2$,
$\therefore (x,x) \in R_1 \cup R_2, \therefore R_1 \cup R_2$ is reflexive.

Symmetric: Let $(x,y) \in R_1 \cup R_2$, then $(x,y) \in R_1 or (x,y) \in R_2$
For R_1, R_2 are equivalence relations
If $(x,y) \in R_1, \therefore (y,x) \in R_1$
If $(x,y) \in R_2, \therefore (y,x) \in R_2$
$\therefore (y,x) \in R_1 \cup R_2$

Transitive: Let $(x,y) \in R_1 \cup R_2, (y,z) \in R_1 \cup R_2$
If $(x,y) \in R_1, (y,z) \in R_2$
We can not have $(x,z) \in R_1 \cup R_2$
So transitive is not satisfied.

b) Yes

Reflexive: $\forall x \in S.(x,x) \in R_1, (x,x) \in R_2$,
$\therefore (x,x) \in R_1 \cap R_2, \therefore R_1 \cap R_2$ is reflexive.

Symmetric: Let $(x,y) \in R_1 \cap R_2$, then $(x,y) \in R_1 and (x,y) \in R_2$
For R_1, R_2 are equivalence relations
If $(x,y) \in R_1, \therefore (y,x) \in R_1$

If $(x, y) \in R_2, \therefore (y, x) \in R_2$

$\therefore (y, x) \in R_1 \cap R_2$

Transitive: Let $(x, y) \in R_1 \cap R_2, (y, z) \in R_1 \cap R_2$

$(x, y) \in R_1, (y, z) \in R_1,$ *then* $(x, z) \in R_1$

$(x, y) \in R_2, (y, z) \in R_2,$ *then* $(x, z) \in R_2$

$(x, z) \in R_1 \cap R_2$

So transitive is satisfied.