Discrete Mathematics I

Tutorial 05 - Answer

Refer to Chapter 2.3

1. Why is f not a function from **R** to **R** if
   a) $f(x) = 1/x$? b) $f(x) = \sqrt{x}$? c) $f(x) = \pm\sqrt{x^2+1}$?

Answer:
   a) $f(0)$ is not defined
   b) $f(x)$ is not defined for $x<0$
   c) Two different values are assigned to each x

2. Determine whether f is a function from **Z** to **R** if
   a) $f(n) = \pm n$? b) $f(n) = \sqrt{n^2+1}$ c) $f(n) = 1/(n^2 - 4)$.

Answer:
   a) No, two values are assigned to each n
   b) Yes
   c) No, $f(2)$ and $f(-2)$ are not defined

3. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a) the function that assigns to each bit string the number of ones minus the number of zeros
   b) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)
   c) the function that assigns to each positive integer the largest perfect square not exceeding this integer
   d) the function that assigns to each pair of positive integers the maximum of these two integers
   e) the function that assigns to a bit string the number of times the block 11 appears

Answer:
   a) Domain: the set of bit strings
      Range: the set of integer (number of 1s – number of 0s)

b) Domain: the set of bit strings
Range: the set of nonnegative integers not exceeding 7.

c) Domain: the set of positive integer
Range: the set of perfect square

d) Domain: $Z^+ \times Z^+$
Range: Z^+

e) Domain: the set of bit strings
Range: N

4. Determine whether each of these functions from **Z** to **Z** is one-to-one or onto.
a) $f(n) = n - 1$ b) $f(n) = n^2 + 1$ c) $f(n) = n^3$ d) $f(n) = \lceil n/2 \rceil$

Answer:

	One-to-one	Onto
a)	Yes	Yes
b)	No	No
c)	Yes	No
d)	No	Yes

5. Give an example of a function from **N** to **N** that is
a) one-to-one but not onto.
b) onto but not one-to-one.
c) both onto and one-to-one (but different from the identity function).
d) neither one-to-one nor onto.

Answer:
a) $f(x) = x^3$
b) $f(x) = \lceil x/2 \rceil$
c) $f(x) = |x|$
d) $f(x) = 1$

6. Determine whether each of these functions is a bijection from **R** to **R**.
a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$
d) $f(x) = (x^2 + 1)/(x^2 + 2)$

Answer:

	One-to-one	Onto	Bijection
a)	Yes	Yes	Yes
b)	No	No	No
c)	Yes	Yes	Yes
e)	No	No	No

7. Show that the function $f(x) = e^x$ from the set of real number to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is invertible.

Answer:

Counterexample: e^x never can be negative. No coimage for the element of negative real number.

e^x is a strickly increasing function. Therefore, it must be one-to-one. The output of e^x is from 0 to positive infinity.

8. Let $f(x) = 2x$. What is
   a) $f(\mathbf{Z})$?   b) $f(\mathbf{N})$?   c) $f(\mathbf{R})$?

Answer:

a) $f(\mathbf{Z}) = \{\ldots, -6,-4, -2,0,2,4,6,\ldots.\}$
b) $f(\mathbf{N}) = \{0,2,4,6,8,\ldots.\}$
c) $f(\mathbf{R}) = \mathbf{R}$

9. Given $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**. Find
   a) $f \circ g$ b) $g \circ f$ c) $f + g$ d) fg

Answer:

a) $(f \circ g)(x) = x^2 + 4x +5$
b) $(g \circ f)(x) = x^2 + 3$
c) $(f+g)(x) = x^2 + x + 3$
d) $(fg)(x) = x^3 + 2x^2 + x + 2$

10. Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$ b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Answer:

a) $\boldsymbol{f}^{-1}(\boldsymbol{S})$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \cup T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \cup T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \vee \boldsymbol{s} \in T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S\} \cup \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in T\}$

$= f^{-1}(S) \cup f^{-1}(T)$

b) $\boldsymbol{f}^{-1}(\boldsymbol{S})$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \cap T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \cap T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S \wedge \boldsymbol{s} \in T\}$

$= \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in S\} \cap \{\boldsymbol{f}^{-1}(\boldsymbol{s}) \mid \boldsymbol{s} \in T\}$

$= f^{-1}(S) \cap f^{-1}(T)$

11. Show that if x is a real number, then $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

Answer:

According to definition, $x = \lfloor x \rfloor + \varepsilon$, where ε is a real number with $0 \le \varepsilon < 1$

For $x = \lfloor x \rfloor + \varepsilon$

$x - \lfloor x \rfloor = \varepsilon$, imply $\lfloor x \rfloor \le x$,

$x - \varepsilon = \lfloor x \rfloor$, imply $\lfloor x \rfloor = x - \varepsilon > x - 1$

According to definition, $x = \lceil x \rceil - \varepsilon'$, where ε is a real number with $0 \le \varepsilon' < 1$

$x = \lceil x \rceil - \varepsilon'$

$\lceil x \rceil - x = \varepsilon'$, imply $x \le \lceil x \rceil$,

$x + \varepsilon' = \lceil x \rceil$, imply $\lceil x \rceil = x + \varepsilon' < x + 1$

12. Draw graphs of each of these functions.

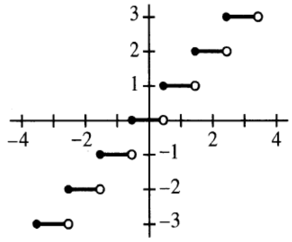
a) $f(x) = \lfloor x + 1/2 \rfloor$ b) $f(x) = \lfloor 2x + 1 \rfloor$

d) $f(x) = \lceil 1/x \rceil$ e) $f(x) = \lceil x - 2 \rceil + \lfloor x + 2 \rfloor$

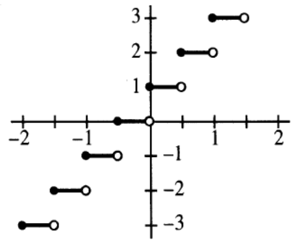
f) $f(x) = \lfloor 2x \rfloor \lceil x/2 \rceil$

Answer:

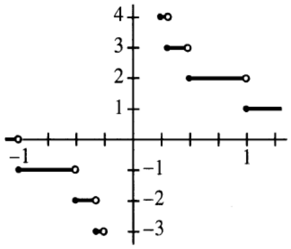
a)



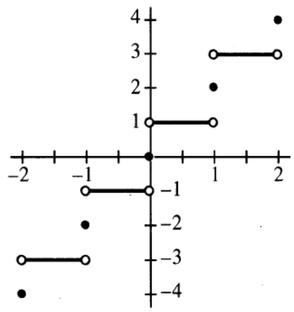
b)



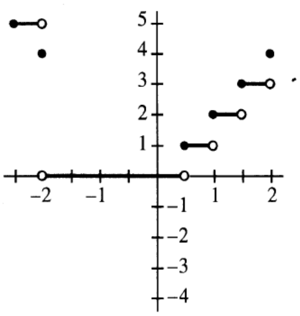
d)



e)



f)



13. Prove or disprove each of these statements about the floor and ceiling functions.
    a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.
    b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.
    c) $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.
    d) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y.

Answer:

a) True.
   Let x = n - ε, where n is an integer and ε is a real number with $0 \leq \leq \varepsilon < 1$
   RHS: $\lceil x \rceil = \lceil n - \varepsilon \rceil = \text{n}$
   LHS: $\lfloor \lceil x \rceil \rfloor = \lfloor \text{ n } \rfloor = \text{n}$

b) False
   Consider LHS: $\lfloor x + y \rfloor = \lfloor \text{ n}_x + \varepsilon_x + \text{n}_y + \varepsilon_y \rfloor$

$2 > \varepsilon_x + \varepsilon_y \geq 0$

2 Cases

Case 1) $2 > \varepsilon_x + \varepsilon_y \geq 1$

LHS: $\lfloor x + y \rfloor = \lfloor n_x + \varepsilon_x + n_y + \varepsilon_y \rfloor$

$= \lfloor n_x + n_y + 1\ (\varepsilon_y + \varepsilon_x - 1) \rfloor$

$= n_x + n_y + 1$

RHS: $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor n_x + \varepsilon_x \rfloor + \lfloor n_y + \varepsilon_y \rfloor = n_x + n_y$

Case 2) $1 > \varepsilon_x + \varepsilon_y \geq 0$

LHS: $\lfloor x + y \rfloor = \lfloor n_x + \varepsilon_x + n_y + \varepsilon_y \rfloor = n_x + n_y$

RHS: $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor n_x + \varepsilon_x \rfloor + \lfloor n_y + \varepsilon_y \rfloor = n_x + n_y$

c) True

Four situations:

Let x = 4n - ε, where n is an integer and ε is a real number with $0 \leq \varepsilon < 1$

LHS: $\lceil \lceil x/2 \rceil /2 \rceil$

$= \lceil \lceil (4n - \varepsilon)/2 \rceil /2 \rceil$

$= \lceil \lceil 2n - \varepsilon/2 \rceil /2 \rceil$

$= \lceil (2n)/2 \rceil$ $\quad (0 \leq \varepsilon/2 < 1)$

$= n$

RHS: $\lceil x/4 \rceil$

$= \lceil n - \varepsilon/4 \rceil$

$= n$ $\quad (0 \leq \varepsilon/4 < 1)$

Let x = (4n-1) - ε, where n is an integer and ε is a real number with $0 \leq \varepsilon < 1$

LHS: $\lceil \lceil x/2 \rceil /2 \rceil$

$= \lceil \lceil (4n-1-\varepsilon)/2 \rceil /2 \rceil$

$= \lceil \lceil 2n - (1+\varepsilon)/2 \rceil /2 \rceil$

$= \lceil 2n/2 \rceil$ $\quad (0 \leq \varepsilon < 1 \rightarrow 1 \leq \varepsilon+1 < 2 \rightarrow 1/2 \leq (\varepsilon+1)/2 < 1)$

$= n$

RHS: $\lceil x/4 \rceil$

$= \lceil (4n-1-\varepsilon)/4 \rceil$

$= \lceil n-(1+\varepsilon)/4 \rceil$

$= n$ $\quad (0 \leq \varepsilon < 1 \rightarrow 1 \leq \varepsilon+1 < 2 \rightarrow 1/4 \leq (\varepsilon+1)/4 < 1/2)$

Let x = (4n-2) - ε, where n is an integer and ε is a real number with $0 \leq \varepsilon < 1$

LHS: $\lceil \lceil x/2 \rceil /2 \rceil$

$= \lceil \lceil (4n-2-\varepsilon)/2 \rceil /2 \rceil$

$= \lceil \lceil 2n - (2+\varepsilon)/2 \rceil /2 \rceil$

$= \lceil (2n-1)/2 \rceil$ $\quad (0 \leq \varepsilon < 1 \rightarrow 2 \leq \varepsilon+2 < 3 \rightarrow 2/2 \leq (\varepsilon+1)/2 < 3/2)$

$= \lceil n-1/2 \rceil$

$= n$ $\quad (0 \leq 1/2 < 1)$

RHS: $\lceil x/4 \rceil$

$= \lceil (4n-2-\varepsilon)/4 \rceil$

$= \lceil n-(2+\varepsilon)/4 \rceil$

$= n$ $\quad (0 \leq \varepsilon < 1 \rightarrow 2 \leq \varepsilon+2 < 3 \rightarrow 2/4 \leq (\varepsilon+2)/4 < 3/4)$

Let $x = (4n-3) - \varepsilon$, where n is an integer and ε is a real number with $0 \le \varepsilon < 1$

LHS: $\lceil \lceil x/2 \rceil /2 \rceil$

$= \lceil \lceil (4n-3-\varepsilon)/2 \rceil /2 \rceil$

$= \lceil \lceil 2n - (3+\varepsilon)/2 \rceil /2 \rceil$

$= \lceil (2n-1)/2 \rceil$ $\quad (0 \le \varepsilon < 1 \rightarrow 3 \le \varepsilon+3 < 4 \rightarrow 3/2 \le (\varepsilon+3)/2 < 2)$

$= \lceil n-1/2 \rceil$

$= n$ $\quad (0 \le 1/2 < 1)$

RHS: $\lceil x/4 \rceil$

$= \lceil (4n-3-\varepsilon)/4 \rceil$

$= \lceil n-(3+\varepsilon)/4 \rceil$

$= n$ $\quad (0 \le \varepsilon < 1 \rightarrow 3 \le \varepsilon+3 < 4 \rightarrow 3/4 \le (\varepsilon+3)/4 < 1)$

d) True

Let $x=n+\varepsilon$, $y=m+\varepsilon'$, where n and m are integers, ε and ε' are a real number with $0\le \le \varepsilon, \varepsilon' < 1$

LHS $= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$

$= \lfloor n+\varepsilon \rfloor + \lfloor m+\varepsilon' \rfloor + \lfloor n+\varepsilon + m+\varepsilon' \rfloor$

$= n + \lfloor \varepsilon \rfloor + m + \lfloor \varepsilon' \rfloor + n + m + \lfloor \varepsilon + \varepsilon' \rfloor$

$= 2n + 2m + \lfloor \varepsilon + \varepsilon' \rfloor$

RHS $= \lfloor 2x \rfloor + \lfloor 2y \rfloor$

$= \lfloor 2n+2\varepsilon \rfloor + \lfloor 2m+2\varepsilon' \rfloor$

$= 2n+2m+\lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor$

Therefore, proof of $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$ is equal to the proof of $\lfloor \varepsilon + \varepsilon' \rfloor \le \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor$

$\lfloor \varepsilon + \varepsilon' \rfloor \le \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor$

Case 1: $0\le \le \varepsilon < 0.5$ and $0\le \le \varepsilon' < 0.5$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 0$

RHS $= \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor = 0$

LHS = RHS

Case 2: $0.5\le \le \varepsilon < 1$ and $0\le \le \varepsilon' < 0.5$

RHS $= \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor = 1$

Case 2.1 : $0\le \le \varepsilon + \varepsilon' < 1$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 0$

Case 2.2 : $1 \le \varepsilon + \varepsilon' < 1.5$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 1$

LHS $\le$ RHS

Case 3: $0 \le \varepsilon < 0.5$ and $0\le.5 \le \varepsilon' < 1$

RHS $= \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor = 1$

Case 3.1 : $0 \leq \leq \varepsilon + \varepsilon' < 1$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 0$

Case 3.2 : $1 \leq \varepsilon + \varepsilon' < 1.5$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 1$

LHS $\leq$ RHS

Case 4: $0.5 \leq \varepsilon < 1$ and $0 \leq .5 \leq \varepsilon' < 1$

LHS $= \lfloor \varepsilon + \varepsilon' \rfloor = 1$

RHS $= \lfloor 2\varepsilon \rfloor + \lfloor 2\varepsilon' \rfloor = 2$

LHS < RHS