Discrete Mathematics I

Tutorial 03 - Answer

Refer to Chapter 1.5 and 1.6

1. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
   a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."
   b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."
   c) "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."
   d) "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

Answer:

a) P(x): I take x off
Q(x): x rains.
R(x): x snows.

$\forall x\,(P(x)\rightarrow(Q(x)\vee R(x)))\quad (P(Tue)\vee P(Thu))\quad (\neg Q(Tue)\wedge\neg R(Tue))\quad (\neg R(Thu))$

$\forall x\,(P(x)\rightarrow(Q(x)\vee R(x)))\wedge(P(Tue)\vee P(Thu))\wedge$
$(\neg Q(Tue)\wedge\neg R(Tue))\wedge(\neg R(Thu))$

$\Rightarrow \underline{\mathbf{(P(Tue)\rightarrow(Q(Tue)\vee R(Tue)))}}\wedge\mathbf{(P(Thu)\rightarrow(Q(Thu)\vee R(Thu)))}\wedge$
$(P(Tue)\vee P(Thu))\wedge\underline{(\neg Q(Tue)\wedge\neg R(Tue))}\wedge(\neg R(Thu))$

$\Rightarrow \underline{\mathbf{\neg P(Tue)}}\wedge(P(Thu)\rightarrow(Q(Thu)\vee R(Thu)))\wedge\underline{(P(Tue)\vee P(Thu))}\wedge(\neg R(Thu))$

$\Rightarrow (P(Thu)\rightarrow(Q(Thu)\vee R(Thu)))\wedge\mathbf{P(Thu)}\wedge(\neg R(Thu))$

$\Rightarrow \mathbf{(Q(Thu)\vee R(Thu))}\wedge\neg R(Thu)$

$\Rightarrow Q(Thu)$

It rains on Thursday

b) P: I eat spicy foods.
Q: I have strange dreams.
R: There is a thunder while I sleep.
$P \rightarrow Q \quad R \rightarrow Q \quad \neg Q$

$$
\begin{aligned}
& (P \rightarrow Q) \wedge (R \rightarrow Q) \wedge \neg Q \\
\Rightarrow\ & \neg Q \wedge (P \rightarrow Q) \wedge \neg Q \wedge (R \rightarrow Q) \\
\Rightarrow\ & \neg P \wedge \neg R
\end{aligned}
$$

I did not eat spicy foods and there is not thunder while I sleep.

c) P(x): x is good for corporations.
Q(x): x is good for the United States.
R(x): x is good for you.
S: you buy a lot of stuff.`

$\forall x(P(x) \rightarrow Q(x)) \quad \forall x(Q(x) \rightarrow R(x)) \quad P(S)$

$$
\begin{aligned}
& \forall x(P(x) \rightarrow Q(x)) \wedge \forall x(Q(x) \rightarrow R(x)) \wedge P(S) \\
\Rightarrow\ & \forall x(P(x) \rightarrow Q(x) \wedge Q(x) \rightarrow R(x)) \wedge P(S) \\
\Rightarrow\ & \forall x(P(x) \rightarrow R(x)) \wedge P(S) \\
\Rightarrow\ & \forall x(P(x) \rightarrow R(x)) \wedge P(S) \\
\Rightarrow\ & (P(S) \rightarrow R(S)) \wedge P(S) \\
\Rightarrow\ & R(S)
\end{aligned}
$$

Buying a lot of stuff is good for you.

d) P(x): x is a rodent.
Q(x): x gnaws their food.
$\forall x(P(x) \rightarrow Q(x)) \quad P(\text{Mice}) \quad \neg Q(\text{Rabbit}) \quad \neg P(\text{Bat})$

$$
\begin{aligned}
& \forall x(P(x) \rightarrow Q(x)) \wedge P(\text{Mice}) \wedge \neg Q(\text{Rabbit}) \wedge \neg P(\text{Bat}) \\
\Rightarrow\ & (P(\text{Mice}) \rightarrow Q(\text{Mice})) \wedge P(\text{Mice}) \wedge \\
& (P(\text{Rabbit}) \rightarrow Q(\text{Rabbit}) \wedge \neg Q(\text{Rabbit}) \wedge \\
& (P(\text{Bat}) \rightarrow Q(\text{Bat}) \wedge \neg P(\text{Bat}) \\
\Rightarrow\ & Q(\text{Mice}) \wedge \neg P(\text{Rabbit}) \wedge \neg P(\text{Bat})
\end{aligned}
$$

Mice gnaw their food. Rabbits are not rodents.

2. For each of these arguments determine whether the argument is correct or incorrect and explain why.
   a) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
   b) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.
   c) Quincy likes all action movies. Quincy likes the movie Eight Men Out. Therefore, Eight Men Out is an action movie.
   d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

Answer:

a) P(x): x is a computer science.
   Q(x): x takes discrete mathematics.

$$\forall x\ (P(x) \rightarrow Q(x)) \wedge Q(\text{Natasha})$$
$$\Rightarrow\ (P(\text{Natasha}) \rightarrow Q(\text{Natasha})) \wedge Q(\text{Natasha})$$

   P(Natasha) cannot be proved
   False

b) P(x): x eats granola every day.
   Q(x): x is healthy.

$$\forall x\ (P(x) \rightarrow Q(x)) \wedge \neg Q(\text{Linda})$$
$$\Rightarrow\ (P(\text{Linda}) \rightarrow Q(\text{Linda})) \wedge \neg Q(\text{Linda})$$
$$\Rightarrow\ \neg P(\text{Linda})$$
   True

c) P(x): x is an action movie.
   Q(x): x is the movie Quincy likes.

   $\forall x(P(x) \rightarrow Q(x))$ Q(Eight Men Out)

$$\forall x(P(x) \rightarrow Q(x)) \wedge Q(\text{Eight Men Out})$$
$$\Rightarrow\ (P(\text{Eight Men Out}) \rightarrow Q\ (\text{Eight Men Out})) \wedge Q(\text{Eight Men Out})$$

   P(Eight Men Out) cannot be proved
   False

d) $P(x)$: x is a lobstermen
$Q(x, y)$: lobstermen x set y dozen traps
$\forall x\, \exists y\, (P(x) \rightarrow Q(x, y))$ P(Hamilton)

$$\forall x\, \exists y\, (P(x) \rightarrow Q(x, y)) \quad \wedge\ P(\text{Hamilton})$$
$$\Rightarrow\ \exists y\, (P(\text{Hamilton}) \rightarrow Q(\text{Hamilton}, y)) \wedge P(\text{Hamilton})$$
$$\Rightarrow\ \exists y\, Q(\text{Hamilton}, y)$$
True

3. Identify the error or errors in this argument that supposedly shows that if $\forall x\, (P(x) \vee Q(x))$ is true then $\forall x\, P(x) \vee \forall x\, Q(x)$ is true.

1.	$\forall x\, (P(x) \vee Q(x))$	Premise
2.	$P(c) \vee Q(c)$	Universal instantiation from (1)
3.	$P(c)$	Simplification from (2)
4.	$\forall x\, P(x)$	Universal generalization from (3)
5.	$Q(c)$	Simplification from (2)
6.	$\forall x\, Q(x)$	Universal generalization from (5)
7.	$\forall x\, P(x) \vee \forall x\, Q(x)$	Conjunction from (4) and (6)

Answer:

Step (3) cannot use simplification since the operator is OR

4. Prove or disprove that the sum of two odd integers is even.

Answer:

Proof:

Assume a, b are two odd integers.
There exists two integers m, n such that $a=2m+1$ and $b=2n+1$
$a + b = 2m+1+2n+1 = 2(m+n+1)$, that means $a + b$ is even

5. Prove or disprove that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$

Answer:

Proof:

$$(x + y \geq 2) \rightarrow (x \geq 1) \vee (y \geq 1)$$

$$\neg((x \geq 1) \vee (y \geq 1)) \rightarrow \neg(x + y \geq 2)$$

$$\Rightarrow \quad ((x < 1) \wedge (y < 1)) \rightarrow (x + y < 2)$$

Assume $x < 1$ and $y < 1$
By adding these two inequalities

$x + y < 1 + 1$
$x + y < 2$

Therefore, the proof is completed

6. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

Answer:

P: You must get a pair of same color socks after three socks are picked
P: You must get at least two socks with the same color after three socks are picked

$\neg P$: We may not get at least two socks with the same color after three socks are picked

If $\neg P$, we need to have more than or equal to three different colors.
Let Q : we need to have more than or equal to three different colors

However, we only can have two (black or blue). Therefore, Q is not correct

By Proof by Contradiction
$\neg P$ is not correct

7. Prove that , $\sqrt[3]{2}$ is irrational.

Answer:

Proof

Assume $\sqrt[3]{2}$ is irrational,

$\sqrt[3]{2} = p / q$, where p and q have no common factors and q is not equal to 0

$\sqrt[3]{2} = p / q$

$2 = p^3 / q^3$

$2q^3 = p^3$

As p^3 is even, p is also even.

Let p = 2m

$2q^3 = (2m)^3$

$q^3 = 4m^3$

As q^3 is even, q is also even.

Both p and q have a factor 2. Contradiction is found

8. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
   a) a proof by contraposition.
   b) a proof by contradiction.

Answer:

a) Proof:

Assume n is odd, n = 2k+1, where k is an integer

$n^3 + 5 = (2k+1)^3 + 5 = 8k^3+4k^2+2k+6 = 2(4k^3+2k^2+k+3)$, where $4k^3+2k^2+k+3$ is an integer

$n^3 + 5$ is even

b) Proof

P(x) : p is even

$\neg P(n^3+5) \rightarrow P(n)$

$\Rightarrow \quad P(n^3+5) \vee P(n)$

$\neg(P(n^3+5) \vee P(n))$

$\Rightarrow \quad \neg P(n^3+5) \wedge \neg P(n)$

Assume n^3+5 is odd and n is odd

Assume n is odd, $n = 2k+1$, where k is an integer
$n^3 + 5 = (2k+1)^3 + 5 = 8k^3+4k^2+2k+6 = 2(4k^3+2k^2+k+3)$, where $4k^3+2k^2+k+3$ is an integer
Contradict with the assumption

9. Prove that if n is an integer, n is even and 3n + 1 is odd are equivalent.

Answer:

Proof: p: n is even. q:3n+1 is odd

a) $p \rightarrow q$
   Assume "n is even" is true, $n = 2k$, where k is an integer
   Therefore, $3n+1 = 3(2k)+1 = 2(3k)+1$, $2n+1$ is odd
b) $q \rightarrow p$
   Assume "n is even" is false, $n = 2k+1$, where k is an integer
   Therefore, $3n+1 = 3(2k+1)+1 = 2(3k+2)$, $2n+1$ is even

10. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of k - 2 and k + 3.

Answer:

Proof

Existence Part

n is an odd integer, $n = 2k + 1$, where k is an integer
$n = 2k + 1 + 2 – 2 = (k - 2) + (k + 3)$

Uniqueness Part

Let $k = r$ be the solution of $n = (k - 2) + (k + 3)$
$(k - 2) + (k + 3) = (r - 2) + (r + 3)$
$2k = 2r$
$k = r$
This means that if $k \neq r$, then $n \neq (k - 2) + (k + 3)$

11. Prove or disprove that if m and n are integers such that mn = 1, then either m = 1 and n = 1, or else m = -1 and n = -1.

Answer:

Proof:

$(mn = 1) \to (n=1 \land m=1) \lor (n=-1 \land m=-1)$

By Proof by Contradiction

$\neg (\neg(mn = 1) \lor ((n=1 \land m=1) \lor (n=-1 \land m=-1))$

$\Rightarrow \neg (\neg(mn = 1) \lor ((n=1 \land m=1) \lor (n=-1 \land m=-1))$

$\Rightarrow (mn = 1) \land \neg ((n=1 \land m=1) \lor (n=-1 \land m=-1)$

$\Rightarrow (mn = 1) \land (\neg((n=1 \land m=1) \land \neg(n=-1 \land m=-1)))$

$\Rightarrow (mn = 1) \land (n\neq1 \lor m\neq1) \land (n\neq-1 \lor m\neq-1)$

For $(n\neq1 \lor m\neq1)$, there are three situations:

1) $n=1$, $m\neq1$: m (1) = 1, and m = 1, contradiction

2) $m=1$, $n\neq1$: n (1) = 1, and n = 1, contradiction

3) $m\neq1$, $n\neq1$:

mn=1, n=1/m, since $m\neq1$, n is not an integer, contradiction

mn=1, m=1/n, since $n\neq1$, m is not an integer, contradiction

Similar to $(n\neq-1 \lor m\neq-1)$

Therefore, $\neg (\neg(mn = 1) \lor ((n=1 \land m=1) \lor (n=-1 \land m=-1))$ is false

12. Use a proof by cases to show that min(a, min(b, c)) = min(min(a, b), c) whenever a, b, and c are real numbers.

Answer:

min(a, min(b, c)) = min(min(a, b), c)

Proof:

(1) $a \le b \le c$, LHS = a, RHS = a

(2) $a \le c \le b$, LHS = a, RHS = a

(3) $b \le a \le c$, LHS = b, RHS = b

(4) $b \le c \le a$, LHS = b, RHS = b

(5) $c \le a \le b$, LHS = c, RHS = c

(6) $c \le b \le a$, LHS = c, RHS = c

Therefore, the proof is finished

13. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Answer:

Proof

Assume there exist integers x and y such that $2x^2 + 5y^2 = 14$

Since $2x^2$ and $5y^2$ must be positive number,

$2x^2 \leq 14$ and $5y^2 \leq 14$

$x^2 \leq 7$ and $y^2 \leq 2.8$

x may be equal to -2, -1, 0, 1, 2

y may be equal to -1, 0, 1

Substitute the largest value of x and y, 2 and 1, to the LHS

$2x^2 + 5y^2 = 2 \text{ x } 4 + 5 = 13 \leq 14$

No solution can be found.